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ON THE NUMBER OF MULTIPLICATIVE PARTITIONS

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I. A Number-Theoretic Function. In this note we show that if $f(n)$ is the number of essentially different factorizations of n, then

$$f(n) \leqslant 2n^{\sqrt{2}}.$$

In considering numbers that have exactly k divisors, one is led to examine this function $f(n)$, the number of ways to write n as the product of integers $\geqslant 2$, where we consider factorizations that differ only in the order of the factors to be the same. We call these representations of n **multiplicative partitions**. For example, $f(12) = 4$, since

$$12 = 6 \cdot 2 = 4 \cdot 3 = 3 \cdot 2 \cdot 2$$

are the four multiplicative partitions of 12. From these four representations, we can conclude that a number has exactly 12 divisors if and only if its prime factorization is one of the following:

$$p^{11}, p^5q, p^3q^2, p^2qr.$$

This follows from the expression for $\tau(n)$, the number of divisors of $n = p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k}$.

$$\tau(n) = \prod_{j=1}^{k} (1 + a_j).$$

For example, see [**1**].

The behavior of $f(n)$ is quite erratic, and apparently has not been previously studied in this form. We observe that if q is prime, then $f(q^k) = p(k)$, the number of additive partitions of k. Also, if $q_1, q_2, \ldots, q_k$ are distinct primes, then $f(q_1q_2 \cdots q_k) = B(k)$, the kth Bell number. See [**2**].

More generally, $f(q_1^{a_1} \cdots q_k^{a_k})$ is the number of additive partitions of the "multi-partite number" $(a_1, a_2, \ldots, a_k)$, where addition is defined component-wise. See [**3**] for further details.

We will show that

$$f(n) \leqslant 2n^{\sqrt{2}}. \tag{1}$$

For a table of $f(n)$ for $1 \leqslant n \leqslant 100$, see the Appendix.

II. Proof of the Main Result. To prove (1) we first define an auxiliary function:

$g(m, n)$ = the number of multiplicative partitions of n with all elements $\leqslant m$.

Clearly $f(n) = g(n, n)$. We have the following

THEOREM 1.

$$g(m,n)=\sum_{\substack{d|n\\ d\leqslant m}} g(d,n/d). \tag{2}$$

Proof. We define $g(m,1)=1$ and $g(1,n)=0$ for $n\neq 1$. Let $n=a_1a_2\cdots a_k$ be a multiplicative partition of n with all factors $\leqslant m$. Then we may assume the factors are arranged in decreasing order, so a_1 is the largest factor in the product. The number of ways to choose $a_2\cdots a_k$ is therefore $g(a_1,n/a_1)$. But a_1 was unspecified, and therefore could be any divisor d of n such that $d\leqslant m$. Summing over all such d gives the result. □

From Theorem 1 we can obtain a simple estimate for $g(m,n)$.

THEOREM 2.

$$g(m,n)\leqslant mn.$$

Proof. The theorem is clearly true for $m=1$ or $n=1$. We will show it is true by induction on the product mn. Assume true for all m, n such that $mn<MN$, where $M\geqslant 2$. Then from Theorem 1 we have

$$g(M,N)=\sum_{\substack{d|N\\ d\leqslant M}} g(d,N/d).$$

Since $d\cdot N/d=N<MN$, we may apply the induction hypothesis to the terms inside the summation. We find

$$\begin{aligned} g(M,N) &\leqslant \sum_{\substack{d|N\\ d\leqslant M}} d\cdot N/d\\ &\leqslant \sum_{d\leqslant M} N\\ &= MN, \end{aligned}$$

and the theorem is true by induction. □

Theorem 2 gives the estimate $f(n)=g(n,n)\leqslant n^2$. It is possible to improve this estimate, which we do in a moment. First we need three easy lemmas.

LEMMA 3.

$$g(a,b)\leqslant g(b,b).$$

Proof. This follows immediately, since if $a\geqslant b$, we have strict equality, while if $a<b$, we have summing over fewer terms of equation (2). □

LEMMA 4. *Let* $0<c<1$. *Then*

$$f(n)\leqslant g(n^c,n)+\sum_{d=1}^{n^{1-c}} f(d).$$

Proof.

$$\begin{aligned} f(n)=g(n,n) &= \sum_{d|n} g(d,n/d)\\ &= \sum_{\substack{d|n\\ d\leqslant n^c}} g(d,n/d)+\sum_{\substack{d|n\\ d>n^c}} g(d,n/d) \end{aligned}$$

$$\leqslant g(n^c, n) + \sum_{\substack{d|n \\ d>n^c}} g(n/d, n/d) \text{ (by Theorem 1 and Lemma 3)}$$

$$= g(n^c, n) + \sum_{\substack{d|n \\ d<n^{1-c}}} g(d, d)$$

$$\leqslant g(n^c, n) + \sum_{d=1}^{n^{1-c}} f(d),$$

which is the desired result. □

LEMMA 5. *Let $a \geqslant 0$. Then*

$$\sum_{d=1}^{k} d^a \leqslant \frac{k^{a+1}}{a+1} + k^a.$$

Proof. This is easily proved by comparison with the integral $\int_1^k t^a \, dt$.

We are now in a position to prove our main result.

THEOREM 6.

$$f(n) \leqslant 2n^{\sqrt{2}}.$$

Proof. The table in the Appendix shows the theorem is true for $n \leqslant 69$. We will prove the theorem by induction on n. Assume $f(d) \leqslant kd^{c+1}$ for $d < n$, where $n \geqslant 70$ and c and k are constants to be specified later. Then from Lemma 4 we have

$$f(n) \leqslant g(n^c, n) + \sum_{d=1}^{n^{1-c}} f(d)$$

$$\leqslant n^{c+1} + \sum_{d=1}^{n^{1-c}} f(d) \text{ (by Theorem 2)}$$

$$\leqslant n^{c+1} + k \sum_{d=1}^{n^{1-c}} d^{c+1} \text{ (by induction)}$$

$$\leqslant n^{c+1} + k\left(\frac{(n^{1-c})^{c+2}}{c+2} + (n^{1-c})^{c+1}\right) \text{ (by Lemma 5)}.$$

Now put $k = 2$ and $c = \sqrt{2} - 1$ to get

$$f(n) \leqslant n^{\sqrt{2}} + \frac{2}{\sqrt{2}+1} n^{\sqrt{2}} + 2n^{2(\sqrt{2}-1)}$$

$$\leqslant 2n^{\sqrt{2}}$$

since $2/\sqrt{2} + 1 < 5/6$ and $2n^{2(\sqrt{2}-1)} \leqslant 1/6 n^{\sqrt{2}}$ for $n \geqslant 70$.

Our theorem is now proved by induction. □

III. Two Conjectures. Numerical evidence seems to indicate that the exponent $\sqrt{2}$ in Theorem 6 is too large. We make two conjectures; the second is more doubtful.

CONJECTURE 1.

$$f(n) \leqslant n.$$

CONJECTURE 2.

$$f(n) \leqslant \frac{n}{\log n} \text{ for } n \neq 144.$$

Both these conjectures have been verified by computer for $n \leqslant 10{,}000$.

Appendix

n	$f(n)$	n	$f(n)$	n	$f(n)$	n	$f(n)$
1	1	26	2	51	2	76	4
2	1	27	3	52	4	77	2
3	1	28	4	53	1	78	5
4	2	29	1	54	7	79	1
5	1	30	5	55	2	80	12
6	2	31	1	56	7	81	5
7	1	32	7	57	2	82	2
8	3	33	2	58	2	83	1
9	2	34	2	59	1	84	11
10	2	35	2	60	11	85	2
11	1	36	9	61	1	86	2
12	4	37	1	62	2	87	2
13	1	38	2	63	4	88	7
14	2	39	2	64	11	89	1
15	2	40	7	65	2	90	11
16	5	41	1	66	5	91	2
17	1	42	5	67	1	92	4
18	4	43	1	68	4	93	2
19	1	44	4	69	2	94	2
20	4	45	4	70	5	95	2
21	2	46	2	71	1	96	19
22	2	47	1	72	16	97	1
23	1	48	12	73	1	98	4
24	7	49	2	74	2	99	4
25	2	50	4	75	4	100	9

References

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2. G. T. Williams, Numbers generated by the function $e^{e^{x}-1}$, this MONTHLY, 52 (1945) 323–327.

3. George Andrews, The Theory of Partitions, Encyclopedia of Mathematics and Its Applications 2, Gian-Carlo Rota, Editor, Addison-Wesley, Reading, Mass. 1976.