TRIPLE POINTS OF IMMERSED $2n$-MANIFOLDS IN $3n$-SPACE

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PROBLEM 4.20 in the Kirby problem list [**5**] asks if there is an immersed 4-manifold in R^6 with a single triple point, in analogy with Boy's surface [**4**], an immersion of RP^2 in R^3. We answer this affirmatively, and generalize the construction to give a $2n$-manifold in $3n$-space with a single triple point. We also give a short proof that no *oriented* 4-manifold can be immersed in R^6 with a single triple point (this proof is due to Gompf [**2**]).

The theorems and their proofs

THEOREM 1. *There is an immersion of* $RP^2 \times RP^2 \# k(S^1 \times S^3)$ *in* R^6 *with one triple point, for some k.*

THEOREM 2. *There is an immersion of* $(\underbrace{RP^2 \times \cdots \times RP^2}_{n\ \text{times}})\# k(S^1 \times S^{2n-1})$ *in* R^{3n} *with one triple point, for some k.*

The proof of Theorem 2 is so strongly analogous to that of Theorem 1 that we leave it to the reader to fill in the necessary details.

Let us establish some notation:

f: $M^n \looparrowright R^{n+k}$ is an immersion,

$e \in H_{n-k}(M; Z_2)$ denotes the Poincaré dual to the Euler class of the normal bundle of f (the 'normal Euler class'),

D: $H^*(M; Z_2) \to H_*(M; Z_2)$ denotes Poincaré duality in M (sometimes with integer coefficients),

$w_i \in H^i(M; Z_2)$ are the Stiefel–Whitney classes of M,

$\bar{w}_i \in H^i(M; Z_2)$ are the normal Stiefel–Whitney classes of M,

Σ_p is the p-tuple set of f in M, i.e. the set of points $x \in M$ such that $f(x)$ is mapped to by p or more points in M. Thus $\Sigma_1 = M$, $\Sigma_2 =$ the preimage of all double values or triple values or higher multiple values, etc.,

$p_1 \in H_4(M; Z)$ is the Pontriagin class of M, and

σ is the signature of the intersection pairing.

Furthermore, let us assume that all immersions are smooth and completely regular, i.e. that their multiple point sets are immersed manifolds in R^n as well, and that they are as self-transverse as they can be (see [**3**] for formal definitions), and that all manifolds are closed, smooth manifolds.

If K is a submanifold of M, $[K]$ denotes the image of the fundamental homology class of K under inclusion into M. The symbol $\cdot$ is used to denote the homology intersection pairing (the dual to cup product).

With these definitions aside, we can proceed with the proofs. Theorem 1 requires a result from [**3**].

THEOREM 3 [**3**]: *If $f\colon M^{2n} \looparrowright R^{3n}$ is a (completely regular) immersion, then*

$$[\Sigma_3] = e \cdot e \in H_0(M; Z_2)$$

This is actually a special case of a far more general theorem. In particular, the result is true with integer coefficients if n is even and M is oriented. In this case, we have (using Corollary 15.8 of [**7**]) that $[\Sigma_3] = e \cdot e = D(p_1(\nu_f)) = -D(p_1(M)) = -3\sigma$ (using Theorem 19.4 of [**7**]). Now the mod 2 reduction of e is $D(w_2(\nu_f)) = D(w_2(M))$; since $w_2 \cup x = x \cup x$ for $x \in H^2(M; Z_2)$ (by Theorem 11.14 *of* 7), e is characteristic for the intersection pairing. By Lemma 5.2 of [**6**], $e \cdot e \equiv \sigma \bmod 8$. Thus $\sigma \equiv -3\sigma \bmod 8$, hence σ is even, so p_1 is even, so the cardinality of Σ_3 is even. Thus the number of triple points in the preimage is even, so the number of triple points in the image is even. We have proved:

THEOREM 4 [**2**]. *An immersion of a closed, oriented 4-manifold in R^6 has an even number of triple points.*

We now continue with the proof of Theorem 1.

LEMMA 1. *There is an immersion of RP^2 in R^3.*

An example is provided by Boy's surface (see [**4**]).

LEMMA 2. *There is an immersion of $RP^2 \times RP^2$ in R^6 with an odd number of triple points.*

Proof. Let $b\colon RP^2 \looparrowright R^3$ denote Boy's surface. Now let

$$B\colon RP^2 \times RP^2 \looparrowright R^6\colon (x, y) \mapsto (b(x), b(y)).$$

B is evidently an immersion. Perturb B to get a generic immersion in R^6. We compute the total Stiefel-Whitney class of the normal bundle of B, ν_B, as follows. Let a and a' denote the generators of $H^1(RP^2)$ in the left and right hand factors, respectively. Then

$$\begin{aligned} w(\nu_B) = w(\nu_{RP^2}) \times (\nu_{RP^2}) &= (1+a) \times (1+a') \\ &= (1 \times 1) + (1 \times a') + (a \times 1) + (a \times a') \end{aligned}$$

In particular, $e = D(w_2(\nu_B)) = D(a \times a')$. Thus

$$\begin{aligned}[\Sigma_3] = e \cdot e &= D((a \times a') \cup (a \times a')) \\ &= D((a \cup a') \times (a \cup a')) \\ &= D([RP^2 \times RP^2]) = 1 \in H_0(RP^2 \times RP^2; Z_2)\end{aligned}$$

Thus Σ_3 consists of an odd number of points. But this is three times the number of triple points in the image (mod 2), so this number must be odd as well. This completes the proof.

We now need the notion of 'adding an ambient 1-handle at the k-tuple set of an n-manifold'. Figure 1 depicts this for the case $k = 2$, $n = 3$, and the ambient space R^3.

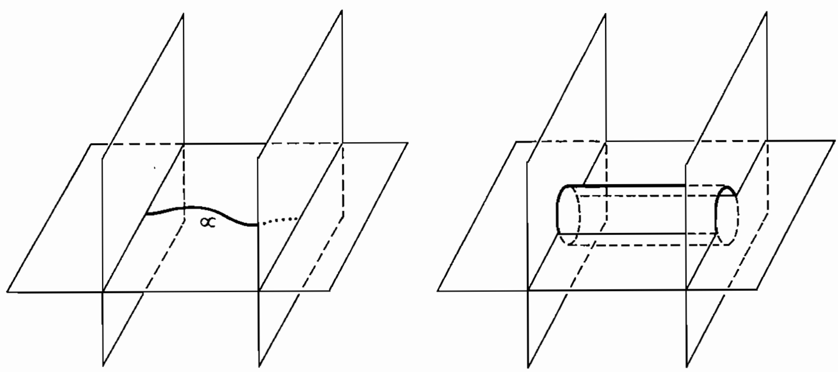

FIG 1

Formally, *adding an ambient 1-handle at the k-tuple set along a curve α* consists of the following: Given a curve α with $\alpha(0)$ and $\alpha(1)$ in Σ_k, and $\alpha(t) \in \Sigma_{k-1}$ for $t \in [0, 1]$, and with each end of α transverse to the sheet it meets at its terminus, we observe that the normal D^n bundles to $\alpha(0)$ and $\alpha(1)$ in these sheets extend to a D^n bundle over all of α, which may be chosen to be generic (i.e. transverse) with respect to the remaining sheets of the immersion in which α lies.

By removing the interior of this D^n bundle over each endpoint of α, and including the boundary of the D^n bundle over *all* of α, we get an immersed $M\#(S^1 \times S^{n-1})$, if M is non-orientable. Note that if M is orientable, the 1-handle might be attached 'wrong'. i.e. by the wrong element of $H_0\ (M; \{\pi_0 O(n)\})$.

Note. One gets, in the construction above, an immersed manifold 'with corners', but these are easily smoothed. One does *not* get a particular *immersion* of $M\#(S^1 \times S^{n-1})$, i.e. one does not get the new *map*, only its image.

This process is done with greater precision and more generality in [**1**].

LEMMA 3. *One may add 1-handles at the double set of an immersion, f, of a 4-manifold M to get a new immersion of a different 4-manifold with connected double set, but the same number of triple points.*

Proof. Let the components of the double set be $D_0, D_1, \ldots, D_p$. Let α_i be a smooth path from D_0 to D_i that misses all the double and triple points of the immersion (since the double set is 2-dimensional and each α_i is 1-dimensional this is easy by transversality). Furthermore, let the α_i's be disjoint, and let each α_i meet D_0 and D_i transversely, away from any triple points. Adding ambient 1-handles along the curves α_i gives the promised immersion.

LEMMA 4. *One may add 1-handles at the triple set of an immersion, f, of a 4-manifold M to get a new immersion of a different 4-manifold with two fewer triple points.*

Proof. Let α be a path in the double set (which we may assume is connected, by the previous lemma) joining two triple points. Adding a 1-handle along α removes these triple points.

Proof of Theorem 1. Take the immersion B: $RP^2 \times RP^2 \looparrowright R^6$ from Lemma 2 and apply the preceeding Lemmas 3 and 4 repeatedly.

Application. Wells [**8**] computes that the unoriented bordism group of immersed n-manifolds in R^{n+k} is $\pi^s_{n+k}(MO(n))$.

Since the number, mod 2, of triple points of an immersed $2n$-manifold is a bordism invariant (the triple points of the bordism form a 1-manifold, whose boundary comprises the triple points in the ends; thus the number of triple points in each end must be the same, mod 2), we have the corollary to Theorem 2:

COROLLARY. *The* $3n$th *stable homotopy of* $MO(n)$ *is non-zero.*

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