ANOTHER PROOF THAT EVERY EVERSION OF THE SPHERE HAS A QUADRUPLE POINT

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Max and Banchoff [2] have shown that every eversion of the sphere $\mathbf{S}^2$ in $\mathbf{R}^3$ has a quadruple point. They do this by showing that twice the Froissart-Morin eversion gives a generator for π_1 of the space of immersions of $\mathbf{S}^2$ which are fixed near the south pole. This involves detailed analysis of that particular eversion.

We present a simpler proof that every eversion has a quadruple point. Here is a sketch: Take any eversion and cap it off to get an immersion of $\mathbf{S}^3$ in $\mathbf{R}^4$. This immersion has normal degree zero. Regular homotopy classes of $\mathbf{S}^3$'s immersed in $\mathbf{R}^4$ are in one-to-one correspondence with $\mathbf{Z} \times \mathbf{Z}$ [4] and the first factor counts the normal degree minus one.

Counting generic quadruple points, mod 2, gives a homomorphism from $\mathbf{Z} \times \mathbf{Z}$ to $\mathbf{Z}_2$. The Froissart-Morin eversion has a quadruple point, so this map is an epimorphism.

The reflection of the standard embedding of $\mathbf{S}^3$ in $\mathbf{R}^4$ has immersion numbers $(-2, 1)$; since it has no quadruple points, the map to $\mathbf{Z}_2$ must take $(-2, 1)$ to zero. The map must therefore take (A, B) to A mod 2. But for any eversion, the normal degree is zero, hence A is -1; thus every eversion has a quadruple point (in fact an odd number of quadruple points). The details follow.

Definition. Let $\mathrm{Imm}(\mathbf{S}^3, \mathbf{R}^4)$ denote the regular homotopy classes of immersions of $\mathbf{S}^3$ in $\mathbf{R}^4$ which agree with the standard embedding on the antarctic region.

Let f be an immersion of $\mathbf{S}^3$ into $\mathbf{R}^4$. Consider the domain $\mathbf{S}^3$ as a subset of $\mathbf{R}^4$ by the standard embedding. Then f extends to an orientation-preserving immersion of a neighborhood of $\mathbf{S}^3$ into $\mathbf{R}^4$.

The differential of this extended f can be used to push forward the standard frame $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4]$ of $\mathbf{R}^4$ at $p \in \mathbf{S}^3$ to a frame (in the image $\mathbf{R}^4$) at $f(p)$. Applying the Gram-Schmidt process, gives an orthonormal frame

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in $\mathbf{R}^4$, hence an element of $\mathbf{SO}(4)$. Thus we get a map from $\mathbf{S}^3$ to $\mathbf{SO}(4)$. The immersion f agrees with the standard embedding near the antarctic, so the south pole gets sent to $\mathbf{I} \in \mathbf{SO}(4)$; thus we get an element of $\pi_3(\mathbf{SO}(4)$. We now have a mapping from $\mathrm{Imm}(\mathbf{S}^3, \mathbf{R}^4)$ to $\pi_3(\mathbf{SO}(4))$ and for every immersion $f: \mathbf{S}^3 \hookrightarrow \mathbf{R}^4$, we let $I(f)$ denote the element of $\pi_3(\mathbf{SO}(4))$ corresponding to the regular homotopy class of f. Under the standard splitting of $\pi_3\mathbf{SO}(4)$ as $\pi_3\mathbf{S}^3 \times \pi_3\mathbf{SO}(3)$, (see below), $I(f) \in \mathbf{Z} \times \mathbf{Z}$, and the two factors of $I(f)$ are called the immersion numbers (or immersion invariants) of f.

Definition. Given two immersions $f, g: \mathbf{S}^3 \to \mathbf{R}^4$, define the *oriented connect sum* of f and g, $f \#_0 g$, as follows: take an outward normal vector to $\mathbf{S}^3$ at the north pole and push it forward by the differential of the extended maps of f and g. Connect the images of f and g by an untwisted tube which leaves the images in the directions specified by the vectors. The result is the image of a new immersion of $\mathbf{S}^3$ in $\mathbf{R}^4$, and this immersion is $f \#_0 g$. (Notice that the new immersion is not well-defined, but it *is* well-defined up to regular homotopy). See Figure 1 for an example of oriented connect sum for a pair of $\mathbf{S}^1$'s in $\mathbf{R}^2$.

LEMMA 1. *Oriented connect sum defines a group operation on* $\mathrm{Imm}(\mathbf{S}^3, \mathbf{R}^4)$, *and the map* I: $\mathrm{Imm}(\mathbf{S}^3, \mathbf{R}^4) \to \pi_3\mathbf{SO}(4)$ *is an isomorphism of groups.*

Proof. The inverse of an immersion f is given by letting $\bar{f}$ be the composition of f with a reflection through a hyperplane in the domain. Then the oriented connect sum of $\bar{f}$ with an immersion with invariants (2, -1) gives the inverse of f. (See [1] for details.) That I is a homomorphism follows from the definition of the product in $\pi_3\mathbf{SO}(4)$. Smale [4] shows that it is a monomorphism, hence an isomorphism.

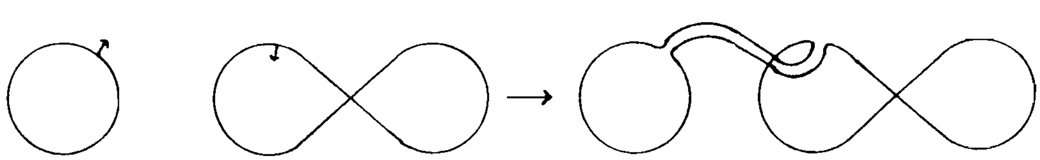

Figure I.

Next we recall the standard splitting of $\mathbf{SO}(4)$ as $\mathbf{S}^3 \times \mathbf{SO}(3)$. (Compare [5], p. 117). Consider $\mathbf{S}^3$ as the set of unit quaternions,

$$\mathbf{S}^3 = \{q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \,|\, q\bar{q} = a^2 + b^2 + c^2 + d^2 = 1\},$$

and let $L(q) \in \mathbf{SO}(4)$ denote the matrix of left multiplication by q. Define $p_1 : \mathbf{SO}(4) \to \mathbf{S}^3$ by

$$p_1([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)] = \mathbf{v}_1,$$

and define $p_2 : \mathbf{SO}(4) \to \mathbf{SO}(3)$ by

$$p_2([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)] = L(\mathbf{v}_1)^{-1}[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]$$

$$= [1, \bar{\mathbf{v}}_1\mathbf{v}_2, \bar{\mathbf{v}}_1\mathbf{v}_3, \bar{\mathbf{v}}_1\mathbf{v}_4].$$

Evidently $p_2(A)$ is a matrix with a 1 in the upper left-hand corner, hence $p_2(A)$ lies in an $\mathbf{SO}(3)$ subgroup of $\mathbf{SO}(4)$.

If $\alpha \in \pi_3\mathbf{SO}(4)$, then $p_{1*}(\alpha) \in \pi_3\mathbf{S}^3$ and $p_{2*}(\alpha) \in \pi_3(\mathbf{SO}(3))$ completely determine α. The element $I(f) \in \pi_3\mathbf{SO}(4)$ will henceforth be decomposed into $(I_1(f), I_2(f)) \in \pi_3(\mathbf{S}^3) \times \pi_3\mathbf{SO}(4)$.

LEMMA 2. *If $f : \mathbf{S}^3 \looparrowright \mathbf{R}^4$, and $I(f) = (I_1(f), I_2(f))$, then $I_1(f)$ is one less than the normal degree of f, i.e. the degree of the normal Gauss map from $f(\mathbf{S}^3)$ to $\mathbf{S}^3$.*

Proof. This follows by a direct computation from the definition above.

LEMMA 3. *The reflection of the standard embedding in the $y-z-w$ plane has immersion numbers* $(-2, 1)$.

Proof. Although the reflection does not agree with the standard embedding on the antarctic, it is evidently regularly homotopic to an immersion which does. One can therefore compute the immersion numbers of this embedding from the definitions above, begin careful to choose an extension which is orientation preserving. Such an extension is quaternionic inversion, $q \to \bar{q}/\|q\|^2$. Writing $q = a + bi + cj + dk$, the differential of this is, after restriction to $\mathbf{S}^3$,

$$\begin{bmatrix} 1-2a^2 & -2ab & -2ac & -2ad \\ 2ab & -1+2b^2 & 2bc & 2bd \\ 2ac & 2cb & -1+2c^2 & 2cd \\ 2ad & 2bd & 2dc & -1+2d^2 \end{bmatrix}.$$

Projection to the first column of this matrix sends the unit quaternion q to $-q^{-2}$ and this map of $\mathbf{S}^3$ to $\mathbf{S}^3$ has degree -2, one less than the normal degree of the normal Gauss mapping of the immersion, as anticipated. Thus $I_1 = -2$.

After considerable effort we may determine p_2 of the matrix above to obtain

$$\begin{bmatrix} 2b^2+2a^2-1 & 2bc-2ad & 2bd+2ac \\ 2ad+2bc & 2c^2+2a^2-1 & 2cd+2ab \\ 2bd-2ac & 2cd+2ab & 2d^2+2a^2-1 \end{bmatrix}$$

which represents the generator of $\pi_3\mathbf{SO}(3)$ (see [5], for example). Thus the invariants of the immersion are $(-2, 1)$.

LEMMA 4. *If $f:\mathbf{S}^2 \times I \to \mathbf{R}^3 \times I$ is an eversion and is capped off to give an immersion $F:\mathbf{S}^3 \looparrowright \mathbf{R}^4$, then the first immersion invariant of F must be* -1.

Proof. Since the normal vector to the immersion never points in the $(0, 0, 0, 1)$ direction (it could do so only on the caps, and they both face 'down'), the normal degree is zero. Lemma 2 then applies.

LEMMA 5. *The number of quadruple point of an eversion*, mod 2, *is an invariant of the regular homotopy class of the eversion.*

Proof. A regular homotopy of eversions gives an immersion of $(\mathbf{S}^2 \times \mathbf{I}) \times \mathbf{I}$ into $(\mathbf{R}^3 \times \mathbf{I}) \times \mathbf{I}$ which is an embedding on $(\mathbf{S}^2 \times \{0, 1\}) \times \mathbf{I}$. The quadruple points of this immersion form an immersed 1-manifold, whose boundary points are the quadruple points of the ends. Since every 1-manifold has an even number of boundary points, the result follows.

We need one final fact: there is an eversion [3] with a single quadruple point. We are now prepared to prove

THEOREM 1. *Every eversion of* $\mathbf{S}^2$ *in* $\mathbf{R}^3$ *has a quadruple point.*

Proof. We will assume all eversions have been capped off to give immersions of $\mathbf{S}^3$ in $\mathbf{R}^4$. Counting quadruple points, mod 2, gives a homomorphism from $\mathbf{Z} \times \mathbf{Z}$ to $\mathbf{Z}_2$. Evidently this homomorphism is an epimorphism, since some eversion has a quadruple point, which yields an immersed $\mathbf{S}^3$ with a quadruple point. Furthermore, $(-2, 1)$ is in the kernel, since the reflection of the standard embedding has no quadruple point. The homomorphism must be $(A, B) \mapsto pA + qB$ mod 2, and we know that $p(-2) + q(-1) = 0$ mod 2. Thus q must be zero, and (since the map is an epimorphism), p must be 1. Since every eversion yields a sphere with invariants $(A, B) = (-1, n)$ for some n, we see that every eversion must have a quadruple point.

Remark. Implicit in this discussion is a map from Imm($\mathbf{S}^3$, $\mathbf{R}^4$) to **SI(3, 1)**, the oriented bordism group of immersed 3-manifolds in $\mathbf{R}^4$, which is known to be isomorphic to $\pi_7(\mathbf{S}^4) = \mathbf{Z}_{24}$ = the 3-stem. The map consists of simply considering a 3-sphere as a 3-manifold. The map from $\pi_3\mathbf{SO}(4)$ to $\pi_7(\mathbf{S}^4)$ is closely related to the J-homomorphism.

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