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IMMERSIONS OF SURFACES IN 3-MANIFOLDS

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INTRODUCTION

THE problem of the classification of immersions of submanifolds of codimension at least one up to regular homotopy was reduced to a homotopy theoretic problem by the methods of Hirsch and Smale [6]. These techniques however are awkward to apply. James and Thomas [7] used the Hirsch-Smale results to enumerate the immersions of a surface into $\mathbb{R}^3$. Their technique was non-constructive.

In §1 we will give a classification of immersions of a surface, F, into a 3-manifold, M, in a given homotopy class, up to regular homotopy. It will be shown that there is a correspondence between $H^1(F;\mathbb{Z}_2)$ and the regular homotopy classes homotopic to any given map $f_0: F \to M$. This correspondence is 1-to-1 unless both F and M are non-orientable, in which case it may be 1-to-2. The classification leads to an explicit construction of all regular homotopy classes of maps within a given homotopy class of maps. We carry out this construction in §2.

In §3 we study exactly what occurs during the process of (generic) regular homotopy. This allows us to define various invariants of the regular homotopy class of an immersion. One such invariant allows us to state precisely which regular homotopy classes can be realized by embeddings in $\mathbb{R}^3$. We solve the problem of when regular homotopy classes can be realized without triple points. Finally we obtain a new proof of a theorem of Banchoff on the number of triple points of immersed surfaces in Euclidean space.

We would like to thank Larry Taylor and the referee for their helpful suggestions.

PRELIMINARIES

We will use the following notation:

M denotes a smooth 3-dimensional manifold, not necessarily orientable or closed. F denotes a smooth closed surface. TM denotes the tangent bundle of M. If $f: F \to M$ is a smooth map then Df denotes the induced map on the tangent bundles. Points in a bundle E are denoted by (x, v) where x is a point in the base and v is a point in E_x, the fiber over x.

$[M, N]$ denotes the homotopy classes of maps of M into N. $[f]$ denotes the class of maps homotopic to f. $\mathrm{Bun}_{[f]}(TF, TM)$ is the set of homotopy classes of bundle homomorphisms of the tangent bundle of F to the tangent bundle of M which cover a given map $f: F \to M$ and which are injective on each fiber. $\mathrm{Bun}_{[f]}(TF, TM)$ is the set of homotopy classes of bundle homomorphisms of the tangent bundle of F to the tangent bundle of M which cover a map homotopic to f, and which are injective on each fiber.

An immersion of F into M is a smooth map of rank two at each point x of F. If f_0 and f_1 are immersions of F into M, a regular homotopy of f_0 to f_1 is a map $H: F \times I \to M$ such that $H(x, 0) = f_0(x)$, $H(x, 1) = f_1(x)$ for all x in F, $H(F, t)$ is an immersion for each t, and $DH: TF \times I \to TM$ is a smooth map. We let $\mathrm{Imm}_{[f]}(F, M)$ denote the set of regular homotopy classes of immersions of F into M in a given homotopy class $[f]$.

§1 THE CLASSIFICATION THEOREM

We now state and prove the main theorem.

THEOREM 1.1. *Let F be a closed surface and M be a 3-manifold. Let $[f]$ be a homotopy class of maps of F into M. Then there is a correspondence*

$$Imm_{[f]}(F, M) \leftrightarrow H^1(F; \mathbb{Z}_2)$$

which is 1-1 unless both F and M are non-orientable, in which case it may be 1-to-2.

Proof. The proof will proceed by establishing a chain of correspondences.

i. There is a 1-1 correspondence

$$\mathrm{Imm}_{[f]}(F, M) \leftrightarrow \mathrm{Bun}_{[f]}(TF, TM).$$

This is a consequence of the immersion theory of Hirsch and Smale [6]. This correspondence associates to an immersion f its differential Df.

LEMMA 1.2. *Let $f: F^2 \to M^3$ be a map. Then f is homotopic to an immersion.*

Proof. Papakyriakopoulos [10] shows that f is homotopic to a general position map with simple branch points, i.e. points a neighborhood of which resembles a cone on a figure eight (see Fig. 1).

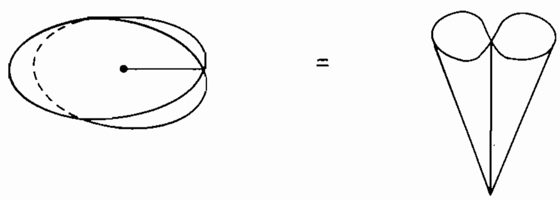


Fig. 1.

Since F is a closed surface, the singularity set (see [5]) consists of closed curves (whose images are double curves) and segments joined at the preimage of a branch point. See Fig. 2.

There is a standard procedure for cutting and pasting to push the branch point along a double curve emanating from it (in the image). See [8] and Fig. 3. This procedure does not alter the homotopy class of the map.

The double curves in the image which are images of segments must end at distinct branch points (see [5] again); by pushing the branch points towards one another along the double curve we can cancel pairs of branch points as in Fig. 3.

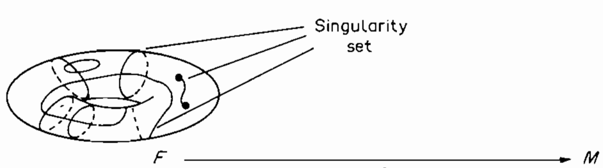


Fig. 2.

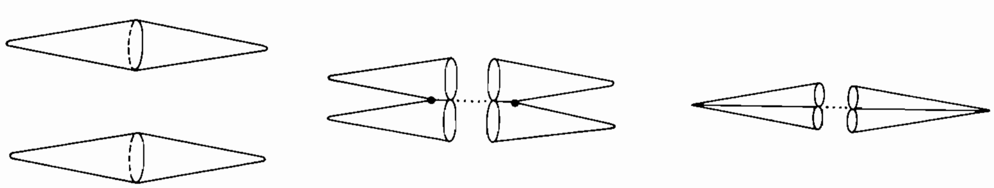


Fig. 3.

Notice that the cancellation takes place within a ball, so that we may draw a standard model for the cancellation as in Fig. 4.

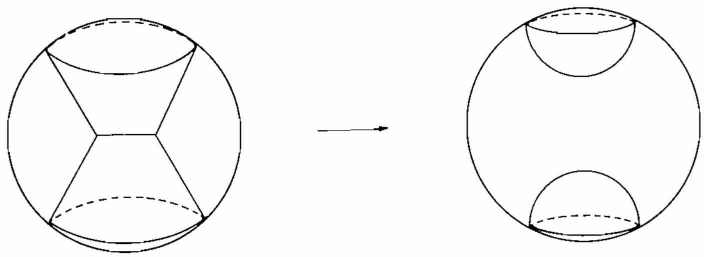


Fig. 4.

When cancellation occurs, the homotopy class of the map *does* change (the domain changes homotopy type), but by adding a small tube with the correct orientation we may recover the original domain and homotopy class. (Each map of $\partial(S^1 \times I)$ to ∂B^3 admits only one extension to a map from $(S^1 \times I)$ to B^3, up to homotopy.)

When we have cancelled all such "double segments" we are left with an immersion of F. It follows that there are immersions in each homotopy class of maps from a surface to a 3-manifold. Now let f_0 be a fixed immersion in the homotopy class $[f]$.

We require a preliminary definition and lemma.

DEFINITION. *A projective framing on an n-dimensional vector bundle is locally a standard frame, and globally continuous modulo multiplication by* $-I$, *where* I *denotes the identity map in* $SO(n)$.

We will show the existence of a projective framing on a non-orientable 3-manifold, which is apparently not a well-known result.

LEMMA 1.3. *Any* 3-*dimensional manifold* M *admits a projective framing.*

Proof. If M is orientable, then it admits a framing. If M is non-orientable, then $w_1(M) \neq 0$. Let V be the line bundle over M with $w_1(V) = w_1(M)$. We will show that the tangent bundle of M is isomorphic to $V \oplus V \oplus V$.

To do this we consider the 3-plane bundles E_1, E_2 over M with $E_1 = TM \otimes V$ and $E_2 = (V \oplus V \oplus V) \otimes V$. Now $E_1 \cong E_2$ iff $TM \cong V \oplus V \oplus V$, as tensoring with a line bundle is a one-to-one map of the set of 3-plane bundles over M to itself. The Wu formula shows that $w_2(M)$ is $w_1(M)^2$, and the Whitney product formula show that

$$w_1(V \oplus V \oplus V) = w_1(M) \quad \text{and}$$

$$w_2(V \oplus V \oplus V) = w_1(M)^2.$$

Thus the first two Stiefel–Whitney classes of M and $V \oplus V \oplus V$ are the same. These determine the first two Stiefel–Whitney classes of E_1 and E_2, [9, p. 89] and E_2 is clearly the trivial bundle over M, so that $w_1(E_1) = w_2(E_1) = 0$. It follows by a standard argument that E_1 is trivial: namely $w_2(E_1) = 0$ implies the existence of a 2-frame over the 2-skeleton of M; $w_1(E_1) = 0$ then gives that this can be made into an oriented 3-frame in a unique way. Finally, there is no obstruction to extending over the 3-cells of M as $\pi_2(0(3)) = 0$.

Thus $E_1 \cong E_2$ and we have proved the Lemma.

As an immediate consequence of this, note that on the tangent bundle to a 3-manifold there is a flat connection (not necessarily torsion free) with parallel translation around a loop always inducing the identity or negative identity on each fiber. To see this, note that TM is the direct sum of a line bundle with itself three times, and that this line bundle has a flat connection, hence M does. Similarly, the holonomy of this line bundle is either the identity or negative identity, so the conclusion follows for TM. The holonomy is the negative identity precisely on orientation reversing curves.

ii. There is a correspondence

$$\text{Bun}_{[f]}(TF, TM) \leftrightarrow \text{Bun}_{f_0}(TF, TM)$$

which is 1-1 if F or M is oriented, and which may be 1-to-2 if both are non-orientable. A necessary condition for the map to be 1-to-2 is that there be an orientation reversing loop $\gamma \in \pi_1(M)$ such that γ commutes with every element of $f_* \pi_1(F)$.

To see this, let $H: F \times I \to M$ be a smooth homotopy of f to f_0. Then for $x \in F$, $\alpha(t) = H(x, t)$ is a path from $f(x)$ to $f_0(x)$. Using the flat connection described above, we have that parallel translation along this path induces a map from $T_{f(x)}M$ to $T_{f_0(x)}M$. Given an element of $\text{Bun}_{[f]}(TF, TM)$, one may compose with this parallel translation (for each $x \in F$) to get an element of $\text{Bun}_{f_0}(TF, TM)$.

Choosing a different homotopy might yield a different element in $\text{Bun}_{f_0}(TF, TM)$. In particular, the difference of two such homotopies (i.e. running one backwards followed by the other forwards) gives a homotopy of f_0 to itself. Parallel transport along such a hoootopy induces (1) the identity or (2) the negative identity on each fiber. Note that by continuity, the homotopy either induces the identity on all fibers or the negative identity on all fibers.

If M is orientable, the holonomy is always the identity, so that the second case never arises.

If F is orientable, the map on TF which is the negative identity on each fiber is homotopic to the identity bundle map, for there is a well-defined sense of clockwise rotation in each fiber (given by the orientation), and rotation by angles of 0 through π gives the desired homotopy.

If neither F nor M is orientable, the correspondence may be 1-to-2, since the identity map is not necessarily homotopic to the negative identity bundle map.

In this case, to induce the negative identity, there must be a homotopy H from f_0 to itself such that for $x_0 \in F$, $\gamma(t) = H(x_0, t)$ is an orientation reversing curve. But let α be any curve on F based at x_0. Then $\gamma\alpha\gamma^{-1} \sim \alpha$, the homotopy being given by the restriction of H to the image of α.

Remark. The case where the map is 1-to-2 can be realized. Consider the manifold $M = D^2 \times I/(x, y; 0) \sim (x, -y; 1)$, the "solid Klein bottle". Let $f: \mathbb{R}P^2 \to M$ be an immersion of the "right-handed" Boy's surface in some 3-ball in M (choose an orientation for the 3-ball to make sense of "right-handed"). By passing this immersion around the I-factor of M, we come back to the left-handed Boy's surface, showing these these two to be the same. Later we will show that these are not regularly homotopic immersions of $\mathbb{R}P^2$in $\mathbb{R}^3$.

To illustrate the π_1 condition, consider $M =$ the non-orientable I-bundle over the torus and $F =$ the Klein bottle, where the meridian is the curve in the torus over which the bundle is not twisted, and the longitude of the Klein bottle is an orientation reversing curve. Immerse F by placing a longitude of F around the meridian of the torus. Then sliding this immersion around the longitude of the torus induces a composition with the negative identity bundle map. Figure 5 gives schematic pictures of these.

The inclusion $\text{Bun}_{f_0}(TF, TM) \subset \text{Bun}_{[f]}(TF, TM)$ is a homotopy inverse to the cor-

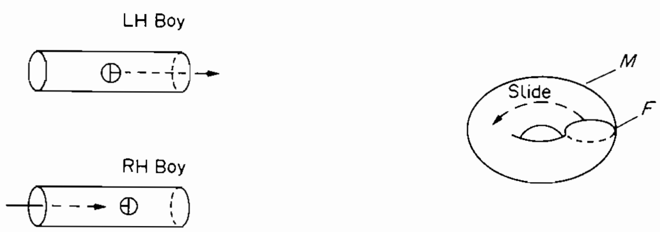


Fig. 5.

respondence in the other direction, when it is 1-to-1, and a homotopy inverse on one component when it is 2-to-1.

Now let $E = f_0^* TM$. Then the differential of f_0 gives a splitting of E as $TF \oplus N$, where N is the normal bundle of f_0 (see [9], page 31). We will start with an element of $\mathrm{Bun}_{f_0}(TF, TM)$ and construct an element of $\mathrm{Bun}_{f_0}(E, TM)$. In constructing this element, we will build maps on TF and N and take their direct sum to give a map on E. Since E splits as a direct sum of TF and N in many ways, we will fix the splitting above and call it the standard splitting.

Notice that if $K \subset E$ is a subbundle of E isomorphic to TF, then (placing a metric on E), its orthogonal complement, $K^\perp$, is isomorphic to N. The proof goes as follows: we have that $TF \oplus N = K \oplus K^\perp$. Letting L be an inverse of TF we get

$$L \oplus TF \oplus N = L \oplus K \oplus K^\perp, \quad \text{so}$$

$$\varepsilon^s \oplus N = \varepsilon^s \oplus K^\perp.$$

Thus N and $K^\perp$ have the same first Stiefel–Whitney class, hence are isomorphic line bundles over F.

Given an element $h \in \mathrm{Bun}_{f_0}(TF, TM)$, define a bundle monomorphism g making the following diagram commute

$$\begin{array}{ccc} & E & \\ {\scriptstyle g}\nearrow & & \searrow{\scriptstyle f^\#} \\ TF & \xrightarrow[h]{} & TM \end{array}$$

(Here $f^\#$ denotes the bundle map in the pullback diagram below):

$$\begin{array}{ccc} E = f_0^* TM & \xrightarrow{f^\#} & TM \\ \downarrow & & \downarrow \\ F & \xrightarrow{f_0} & M \end{array}$$

The map g can be defined fiber-by-fiber by noting that h is a monomorphism and $f^\#$ is an isomorphism on each fiber, so that on each fiber, $f^{\# -1}$ makes sense. Then we can define $g(x, v)$ to be $f^{\# -1}(h(x, v))$.

Since g is a bundle monomorphism, $g(TF)$ is isomorphic to TF, hence $g(TF)^\perp$ is isomorphic to N (see above). Place a metric on each bundle and let $k\colon N \to g(TF)$ be a length-preserving isomorphism (see below for a rule to specify a choice of k). Then $f^\# \circ k$ is a bundle map from N into TM. We can use this to define an element of $\mathrm{Bun}_{f_0}(E, TM)$, namely $h \oplus (f^\# \circ k)$ (this is actually defined on $TF \oplus N$, which we identify with E via the standard splitting).

To specify the choice of the bundle map k, we will fix, once and for all, metrics on TF, E, and TM, and fix a point $x_\circ \in F$, an orthonormal frame $[v_1, v_2]$ of $T_{x_\circ} F$, an orientation of $T_{f_0(x_\circ)} M$, and an orientation of $E_{x_\circ}$ such that $f^\#$ is orientation preserving at $x_\circ$.

Then $v_1 \times v_2 \in E_{x_\circ}$ makes sense, and it is an element of $N_{x_\circ} \subset E_{x_\circ}$. Since k is to be a length preserving isomorphism of a line bundle over a connected space, it is uniquely determined by its value at a single point. We therefore choose to insist that

$$k(v_1 \times v_2) = h(v_1) \times h(v_2)$$

This ensures that the element of $\mathrm{Bun}_{f_0}(E, TM)$ we have constructed lies in $\mathrm{Bun}_{f_0}(E, TM)_\circ$ the set of bundle monomorphisms from E to TM, covering f_0 and preserving the orientation convention at the point $x_\circ$.

We now see that

iii. There is a 1-1 correspondence

$$\mathrm{Bun}_{f_0}(TF, TM) \leftrightarrow \mathrm{Bun}_{f_0}(TF, TM)_\circ$$

Similarly we let $\mathrm{Bun}_{id}(E, E)_{\circ}$ represent those bundle monomorphisms which take the fiber $N_{x_{\circ}}$ to itself by the identity, and which preserve the orientation at $x_{\circ}$. (Henceforth a subscript of "$\circ$" on any set of bundle maps will indicate that a similar condition holds).

We now prove

iv. There is a 1-1 correspondence

$$\mathrm{Bun}_{f_0}(E, TM)_{\circ} \,\mathrm{C}\, \mathrm{Bun}_{id}(E, E)_{\circ}$$

Since $E = f_0^*(TM)$, there is for each point x in F a map $\phi_x \colon T_{f_0(x)}M \to E_x$ which is an isomorphism of vector spaces (it is, at each point, an inverse to $f^{\#}$). Given any map h in $\mathrm{Bun}_{f_0}(E, TM)_{\circ}$, we can compose $h|E_x$ with ϕ_x to get an isomorphism from E_x to itself which depends continuously on x. Thus h induces a map in $\mathrm{Bun}_{id}(E, E)$. The condition on orientations that $f^{\#}$ be orientation preserving at $x_{\circ}$ ensures that the map is actually in $\mathrm{Bun}_{id}(E, E)_{\circ}$. An inverse to this construction proceeds as follows: take a mapping k in $\mathrm{Bun}_{id}(E, E)_{\circ}$ to $f^{\#} \circ k$ in $\mathrm{Bun}_{f_0}(E, TM)$, where $f^{\#}$ is the pullback isomorphism from above.

Note that if M is orientable, then it is parallelizable, and hence E is the trivial bundle. If M is not orientable, its orientation double cover $\tilde{M}$ is parallelizable.

v. If E is trivial then there exists a 1-1 correspondence

$$\mathrm{Bun}_{id}(E, E)_{\circ} \leftrightarrow [F, SO(3)]$$

By fixing a trivialization on E we get a 1-1 correspondence of $\mathrm{Bun}_{id}(E, E)_{\circ}$ with $\mathrm{Bun}_{id}(F \times \mathbb{R}^3, F \times \mathbb{R}^3)_{\circ}$. (Here the subscript of "$\circ$" indicates that the map is orientation preserving on each fiber). Let γ be an element of $\mathrm{Bun}_{id}(F \times \mathbb{R}^3, F \times \mathbb{R}^3)_{\circ}$. Then define δ in $[F, SO(3)]$ by $\delta(x)v = \gamma(x, v)$ for $(x, v) \in F \times \mathbb{R}^3$, and similarly in the other direction.

If E is not trivial, the situation is more complex.

Let $\tilde{f} \colon \tilde{F} \to \tilde{M}$ be the orientation double cover of f_0, i.e. a map such that the diagram

$$\begin{array}{ccc} \tilde{F} & \xrightarrow{\ \tilde{f}\ } & \tilde{M} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ F & \xrightarrow{\ f_0\ } & M \end{array}$$

commutes, where $\tilde{M}$ is the orientation double cover of M, p the standard double covering map, and $\tilde{F}$ and q may *not* be the orientation double cover of F. In particular, if f_0 lifts to a map to $\tilde{M}$, then $\tilde{F} = F$ and q is the identity.

Now let $\tilde{E}$ be the bundle over $\tilde{F}$ which is the pullback by $\tilde{f}$ of $T\tilde{M}$. There is an induced involution σ on $\tilde{E}$, with quotient E. This projects to an involution τ on $\tilde{F}$ with quotient F.

DEFINITION. *A map j in $\mathrm{Bun}_{id}(E, E)_{\circ}$ is in $EQ\mathrm{Bun}_{id}(E, E)_{\circ}$ if $\sigma j = j\sigma$.*

vi. If E is non-orientable there is a 1-1 correspondence

$$\mathrm{Bun}_{id}(E, E)_{\circ} \leftrightarrow EQ\mathrm{Bun}_{id}(\tilde{E}, \tilde{E})_{\circ}$$

Let $\{w_i\}$ be a projective frame on E, as above, which lifts to a real frame $\{v_i\}$ on $\tilde{E}$. Then σ acts on $\tilde{E}$ by

$$\sigma(x, v_i) = (\tau(x), \ -v_i)$$

Let $\tilde{h}$ be a map in $EQ\mathrm{Bun}_{id}(\tilde{E}, \tilde{E})_{\circ}$. Then $\tilde{h}$ gives a map h in $\mathrm{Bun}_{id}(E, E)_{\circ}$ defined by $h(x, v_i) = \pi\tilde{f}(\tilde{x}, \tilde{v}_i)$, where v_i is a projective frame element and $\tilde{x}$ is one of the two points covering x. This is well-defined by the equivariance of $\tilde{h}$, as the alternate choice of lift would give $\pi\tilde{f}(\tau(\tilde{x}), -\tilde{v})$ which equals $\pi\tilde{f}\sigma((\tilde{x}), \tilde{v}) = \pi\tilde{f}((\tilde{x}), \tilde{v})$. The converse is similar.

Now define $EQ\mathrm{Bun}_{id}(\tilde{F} \times \mathbb{R}^3, \ \tilde{F} \times \mathbb{R}^3)_{\circ}$ to be maps, h, in $\mathrm{Bun}_{id}(\tilde{F} \times \mathbb{R}^3, \ \tilde{F} \times \mathbb{R}^3)_{\circ}$ satisfying $h\sigma = \sigma h$, where $\sigma(x, v) = (\tau(x), \ -v)$.

vii. There is a 1-1 correspondence

$$EQ\mathrm{Bun}_{id}(\tilde{E}, \tilde{E})_{\circ} \subset EQ\mathrm{Bun}_{id}(\tilde{F} \times \mathbb{R}^3, \tilde{F} \times \mathbb{R}^3)_{\circ}$$

To see this, use the lift of a projective frame to trivialize $\tilde{F} \times \mathbb{R}^3$.

Define $EQ[F, SO(3)]$ to be maps k in $[F, SO(3)]$ satisfying $k(\tau x) = k(x)$.

viii. There is a 1-1 correspondence

$$EQ\mathrm{Bun}_{id}(\tilde{E}, \tilde{E})_{\circ} \subset EQ[\tilde{F}, SO(3)].$$

The correspondence between g in $EQ\mathrm{Bun}_{id}(\tilde{E}, \tilde{E})_{\circ}$ and γ in $EQ[\tilde{F}, SO(3)]$ is given by

$$\gamma(\tilde{x})v = \pi_2 g(\tilde{x}, v),$$

where π_2 is the projection into the second factor. To see that this is equivariant, note that g equivariant implies $g\sigma = \sigma g$ so that $\gamma(\tilde{x})v = \pi_2 g(\tilde{x}, v)$, and $\gamma(\tau\tilde{x})v = \pi_2 g(\tau\tilde{x}, v)$. Now

$$g\sigma(\tilde{x}, v) = g(\tau(\tilde{x}), -v) = (\tau(\tilde{x}), \gamma(\tau(\tilde{x}))[-v])$$

Using now the equivariance of g, one obtains that

$$\begin{aligned} g\,\sigma(\tilde{x}, v) = \sigma g(\tilde{x}, v) = \sigma(\tilde{x}, \gamma(\tilde{x})[v]) &= (\sigma(x), -\gamma(\tilde{x})[v]) \\ &= (\sigma(x), \gamma(\tilde{x})[-v]) \end{aligned}$$

so that $\gamma(\tau(\tilde{x})) = \gamma(\tilde{x})$ and γ is equivariant.

Similarly, given γ and defining g via $g(x, v) = [x, \gamma(x)v]$, we have

$$\sigma g(\tilde{x}, v) = (\tau\tilde{x}, -\gamma(x)v)$$

while

$$\sigma g(\tilde{x}, v) = g(\sigma(\tilde{x}), -v) = (\tau(\tilde{x}), \gamma(\tau\tilde{x})[-v]) = (\tau(\tilde{x}), -\gamma(\tau\tilde{x})[v])$$

ix. There is a 1-1 correspondence

$$EQ[\tilde{F}, SO(3)] \leftrightarrow [F, SO(3)]$$

The proof of this is standard.

To finish the proof of the theorem we note

x. There is a 1-1 correspondence

$$[\mathrm{F}, SO(3)] \leftrightarrow H^1(F : \mathbb{Z}_2)$$

The proof of this proceeds via obstruction theory. Note that $SO(3)$ is diffeomorphic to $\mathbb{R}P^3$. Since F is a $K(\pi, 1)$, homotopy classes of maps of F to $\mathbb{R}P^3$ are determined by their values on the fundamental group of F. Since the fundamental group of $\mathbb{R}P^3$ is abelian, the homomorphism factors through the Hurewicz homomorphism. It is thus in 1-1 correspondence with maps of the first homology of F to $H_1(\mathbb{R}P^3) = \mathbb{Z}_2$. The result now follows.

Corollary 1.3. (*James–Thomas*) *There are* $2^{2-\chi}$ *distinct regular homotopy classes of immersions of a surface of Euler characteristic* χ *into* $\mathbb{R}^3$.

§2. CONSTRUCTIONS

In this section we will give explicit representatives of each regular homotopy class of immersions of F into M. In the case of T^2 in $\mathbb{R}^3$ these appear in [10]. We begin with four immersions of the annulus and the Mobius band.

Immersion 2.1

$f_1 : S^1 \times D^1 \to S^1 \times D^2$ is obtained by taking the product of the α-immersion of an arc into the 2-disk with S^1.

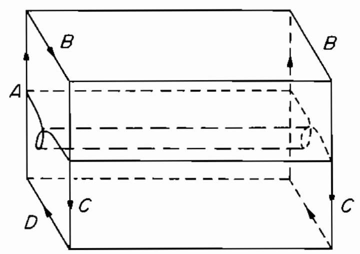


Fig. 6.

Immersion 2.2

Let M denote the Mobius band. Then $f_2: M \to M \times I$ is obtained by identifying $\alpha \times 0$ to $\alpha \times 1$ in $D^2 \times I$ as in the following figure.

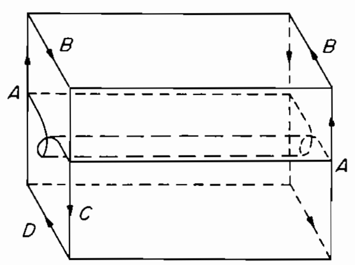


Fig. 7.

Consider now an immersion of D^2 into D^3 whose level sets are given by the following sequence of pictures:

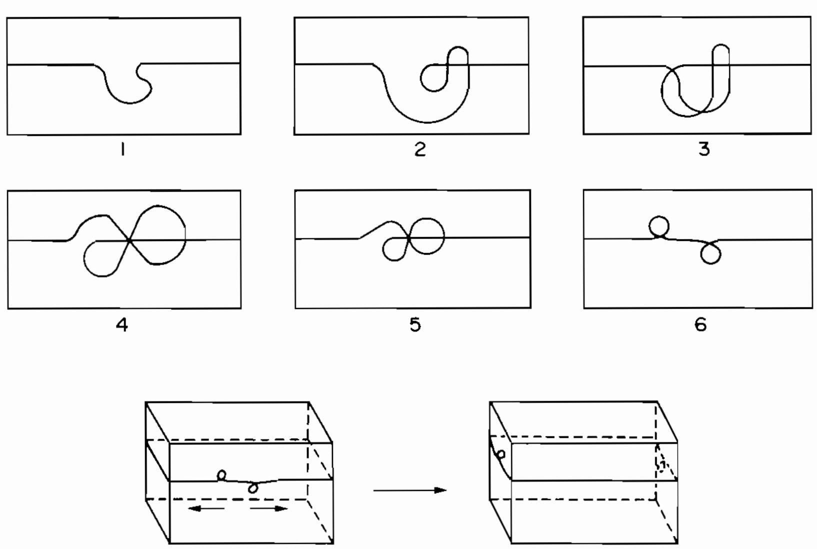


Fig. 8.

We apply the isotopy at the bottom to the trace of this regular homotopy to obtain the "kinky box", an immersion whose boundary looks like the bottom right-hand picture. Note that the kinky box is an immersion of the disk in the 3-ball with a single triple point.

Immersion 2.3

The map $f_3: S^1 \times D^2 \to M^2 \times I$ is obtained by the following identification on the kinky box.

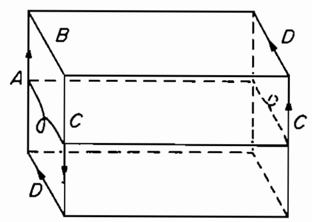


Fig. 9.

Immersion 2.4

The map $f_4: M \times S^1 \to D^3$ is obtained from the kinky box via the following identification.

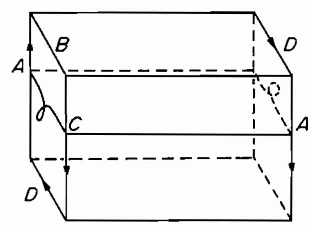


Fig. 10.

We will see that the above four immersions can act as the building blocks to obtain all regular homotopy classes of immersions. Let M be a 3-manifold, let $f: F \looparrowright M$ be a given immersion and let s be a simple closed curve on F. A regular neighborhood of s in F, $N(s)$, is either an annulus or a Möbius band. Moreover, the normal line bundle to $f(N(s))$ is either trivial or non-trivial. We thus have four possible situations.

	Surface 1-sided	Surface 2-sided
Curve 1-sided	Immersion 2.4	Immersion 2.2
Curve 2-sided	Immersion 2.3	Immersion 2.1

In each situation one can alter the map f on $N(s)$, leaving it unchanged elsewhere, in such a way that $N(s)$ is mapped in by one of the f_i. Calling the resulting map g, we say that g is obtained from f by *adding a kink along s*.

Fix, once and for all, an immersion $f: F \looparrowright M$, and suppose given an immersion $f_0: F \looparrowright M$ in the homotopy class of f. By using this f_0 as 'the fixed immersion in the homotopy class $[f]$' in the proof of Theorem 1.1, we see that under the correspondences of that theorem, f_0 is sent to the zero element in $H^1(F; \mathbb{Z}_2)$.

Let α_i, β_i, γ_k be a standard set of generators for $H_1(F;\mathbb{Z}_2)$, so that $\alpha_i\cdot\alpha_i = \beta_i\cdot\beta_i = \alpha_i\cdot\gamma_j = \beta_i\cdot\gamma_j = 0$ and $\alpha_i\cdot\beta_j = \delta_{ij} = \gamma_i\cdot\gamma_j$, where δ_{ij} is the Kronecker delta.

Given ϕ: $H_1(F)\to\mathbb{Z}_2$ we will construct an immersion g which induces ϕ via the correspondences of Theorem 1.1 and which is homotopic to f.

Fix a point p in F. Given any curve $c(t)$ in $\pi_1(F, p)$, let $(v, w, n)_p$ be a frame at the point $f(p)$ with v tangent to c, w normal to v and tangent to $f(F)$, and n normal to $f(F)$. We may associate to the path c a path called c_f in $SO(3)$ by letting $a_f(t)$ = the frame at $c(t)$ gotten by continuous translation of the frame $(v, w, n)_p$ to $c(t)$ along c; then we let $c_f(t) = a_f(t)a_{f_0}(t)^{-1}$ (here the superscript denotes matrix inversion).

The correspondences of Theorem 1.1 take "the immersion f" to "the homomorphism taking $[c]$ to the element of $\mathbb{Z}_2$ corresponding to c_f". If we alter the immersion f so that the image of c, c_{new}, crosses an extra kink, then $I(c_{\text{new}}) = 1 + I(c_{\text{old}})$ in $\pi_1(SO(3))$, where I takes curves to elements of $\pi_1(SO(3))$ via the correspondence of Theorem 1.1.

Thus starting with f_0 we can construct an immersion corresponding to ϕ by proceeding as follows. If $\phi(\alpha_i)$ is not zero we add a kink along β_i, and if $\phi(\beta_i)$ is not zero we add a kink along α_i. If $\phi(\gamma_i)$ is not zero, we add a kink along γ_i. One can now check that the resulting immersion g induces ϕ. Given a fixed f_0, we call the immersion constructed the *standard immersion corresponding to f_0 and ϕ.*

§3. APPLICATIONS

We now consider the question of what can happen to a surface during the course of a generic regular homotopy.

THEOREM 3.1. *Let H_t: $F\to M$ be a regular homotopy in general position such that H_0 and H_1 are general position immersions. Then there exist $0 < t_1 < t_2 \ldots t_n < 1$ such that H_t is a general position immersion and $H_t(F)$ is carried to $H_{t'}(F)$ by an ambient isotopy of M whenever $t_i < t, t' < t_{i+1}$, and such that $H_{t_i-\varepsilon}(F)$ differs from $H_{t_i+\varepsilon}(F)$ by one of the following or their inverses:*

1. *Creation of a new double curve as two sheets meet.*

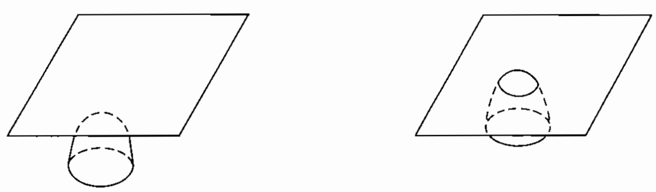


Fig. 11.

2. *A sheet passes a saddle.*

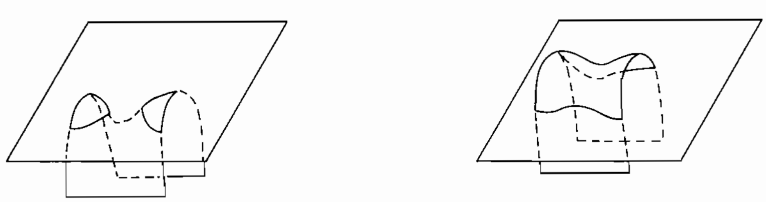


Fig. 12.

3. *A double curve meets a sheet creating two new triple points.*

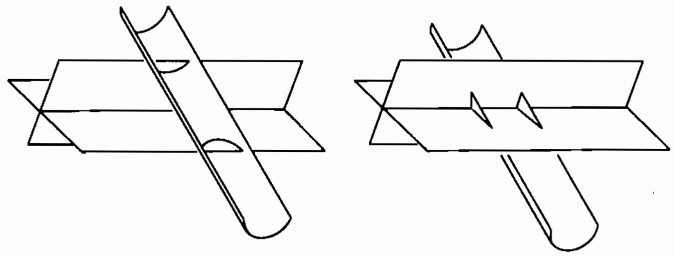


Fig. 13.

4. *A sheet passes through a triple point.*

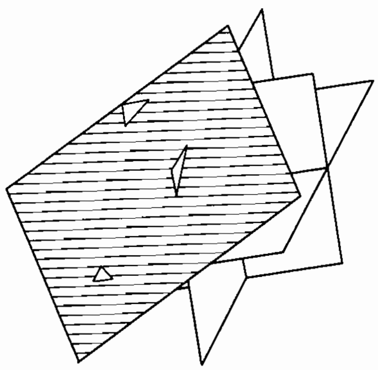


Fig. 14.

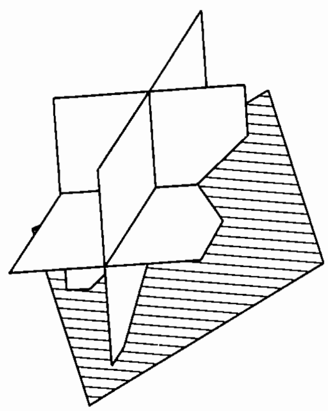


Fig. 15.

Proof. This follows from general position arguments for the map H. See [4] for details.
Given an immersion $f: F \to M$ in general position, define $D_{\text{image}}(f)$ to be the element of $H_1(M\,;\,\mathbb{Z}_2)$ corresponding to all the double curves in the image of f. Define $D_{\text{domain}}(f)$ to be the element of $H^1(F\,;\,\mathbb{Z}_2)$ corresponding to all the preimages of the double curves of f.

PROPOSITION 3.2. *$D_{\text{image}}(f)$ and $D_{\text{domain}}(f)$ are regular homotopy invariants.*

Proof. These are preserved by each process in Theorem 3.1. The following is an example of the use of the invariants above.

COROLLARY 3.3. *Only one of the 4^g regular homotopy classes in $IMM_{[f_0]}(F, F \times I)$ is representable by an embedding, where $f_0(x) = (x, 0)$ and g is the genus of F.*

Proof. We can construct representatives for each regular homotopy class by putting kinks on appropriate curves. In no case except the one with no kinks at all is $D_{\text{image}}(f) = 0$.

PROPOSITION 3.4. *The number of triple points (mod 2) of an immersion is a regular homotopy invariant.*

Proof. This number is preserved by each process in Theorem 3.1.

THEOREM 3.5. *Let f be an embedding. Then every regular homotopy class in $IMM_{[f]}(F, M)$ can be realized without triple points if and only if f is two-sided.*

Proof. If f is two-sided and g is a regular homotopy class of immersions homotopic to f, we can construct a representative of g by adding kinks along appropriate curves, as in Immersions 2.1 and 2.2. If this necessitates adding a kink along two intersecting curves α_i and β_i, then we instead add a kink along the embedded curve $\alpha_i + \beta_i$ to obtain the same regular homotopy class. These kinks introduce no triple points.

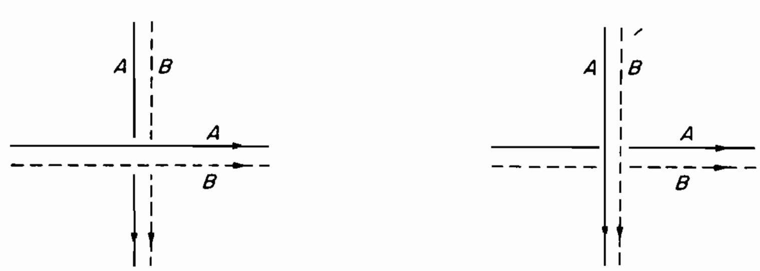
On the other hand, if f is one-sided, we can obtain a distinct regular homotopy class by adding a kink as in Immersions 2.3 or 2.4 along a one-sided embedded curve. This will introduce a single triple point. Corollary 3.2 ensures that some triple point is therefore always present in any representative of this regular homotopy class.

This means that if an embedding is 1-sided we cannot hope to realize all regular homotopy classes homotopic to it by embeddings.

We now consider a quantity associated to a general position immersion f which we call the *twisting invariant* of the immersion, $T(f)$. This is defined as follows. Let A be a double curve of the image of the immersion $f: F \looparrowright \mathbb{R}^3$. In general A is an immersed curve which may have double or triple point self intersections. The preimage of A in F consists either of one immersed curve which double covers its image or two immersed curves which are identified by f. Let a be one of the preimages of A, and let b be a curve on F parallel to a. If a is one sided then b is homotopic to a double-cover of a. Let B be the image of b under f. Then B lies on the image of f nearby to A. We call B a *pushoff of A along f*. B lies on an immersed torus which is a tubular neighborhood of A. This torus can be homotoped to give an embedded torus. On this new embedded torus, both meridian and longitude are well-defined, and B is homotopic to m meridians and l longitudes. We define $t(A, f)$ to be the quotient m/l. The choice of meridian and longitude does not affect this quotient, as long as we fix some orientation convention [12]. We define $T(f)$ to be the sum over all double curves of $t(A, f)$ (mod 2). Since the number of longitudes that B represents on a peripheral torus of A is between one and four, $T(f)$ takes its values in the set $\{0, \frac{1}{4}, \frac{1}{2}, \ldots, 1\frac{3}{4}\}$. We will now show that $T(f)$ is well-defined, and is an invariant of the regular homotopy class of the immersion f. This is a precise definition of an invariant suggested by Eccles [3]. See also [1][2].

LEMMA 3.6. *Let A be a knot in $\mathbb{R}^3$ and let B be a simple closed curve on the peripheral torus of A. Let A' be obtained from A by changing a crossing, and let B' be the corresponding curve on the peripheral torus of A'. If B represents m meridians and l longitudes and B' represents m' meridians and l' longitudes, then $m/l = m'/l'$ (mod 2).*

Proof. Consider a regular projection of A and B, drawn below in the neighborhood of a crossing. To compute m, which is the linking number of A and B, we look at each point where A crosses under B and count as below:



Count −1 for each strand of B

Count +1 for each strand of B

Fig. 16.

Clearly $m' = m \pm 2l$. Since $l' = l$, it follows that $m'/l' = m/l \pmod 2$.

It now follows that $T(f)$ is well-defined, as perturbing a non-embedded double curve in two different ways will change $T(f)$ by an even integer. We will now show that $T(f)$ is invariant under regular homotopy.

THEOREM 3.7. *If f and g are regularly homotopic general position immersions in* $\mathbb{R}^3$, *then* $T(f) = T(g)$.

Proof. We will check that under each of the processes in Theorem 3.1 the quantity $T(f)$ remains unchanged. In the creation of a new double curve or the birth of a pair of triple points, it is easy to see that $T(f)$ remains unchanged. As a sheet passes through a triple point creating a temporary quadruple point, 3 pairs of double curves cross one another as in Lemma 3.6. Thus this lemma shows that $T(f)$ remains unchanged. The most difficult case is when a sheet passes a saddle, as in this case there are many possibilities. We will go through these one by one. The different cases correspond to the different ways that curves a and b below can connect to one another, and to the different possibilities for a pushoff along arc a to connect with a pushoff of arc b to form a pushoff for the resultant system. The different possibilities for the way that a pushoff can permute the four faces coming together along a double curve are the identity, (1, 3) (2, 4), (1, 2, 3, 4) and (4, 3, 2, 1). These correspond to the intersections of a pair of 2-sided annuli, a pair of 1-sided Möbius bands, and a single Möbius band intersecting itself, in a neighborhood of the double curve. Other possibilities cannot occur in an orientable 3-manifold.

1. Arcs a and b lie on distinct double curves, which we will also call a and b. These two double curves then become a single double curve c: We can assume that the orientation of the double curve c agrees with that of a and b. Let m_a, m_b, m_c and l_a, l_b, l_c represent the number of meridians and longitudes of corresponding pushoffs a', b', c'.

i. $l_a = l_b = 1$. In this case, c and c' are as below:

Fig. 17.

Note that b and a cross at an even number of points, by intersection theory for $\mathbb{R}^2$. Each of these points yields a contribution of ± 1 to $lk(c, c')$, so their total contribution is some even integer $2k$. Thus

$$m_c = lk(c, c') = lk(a, a') + lk(b, b') + 2k$$

and

$$\frac{m_c}{l_c} = \frac{m_a}{l_a} + \frac{m_b}{l_b} + 2k \qquad (*)$$

ii. $l_a = 1, l_b = 2$. In this case $l_c = 2$, which can be seen either from the diagram or by taking permutations (1) and (1, 3) (2, 4) to get the resultant permutation. It follows that each of the crossings of a and b gives a contribution of ± 2 to $lk(c, c')$. As there are an even number of such points, they contribute a total of $4k$ for some integer k, so that

$$m_c = lk(c, c') = 2lk(a, a') + lk(b, b') + 4k$$

and $(*)$ holds.

iii. There are several other cases. However from the above, it is clear that in all cases we can compute $lk(c,c')$ by the formula:

$$lk(c,c') = lk(a,a')\frac{l_c}{l_a} + lk(b,b')\frac{l_c}{l_b} + 2k(l_c),$$

from which it follows that $(*)$ holds.

2. a and b are part of a single double curve, which we call a. a and b are oriented as follows:

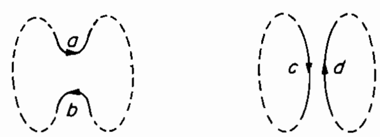


Fig. 18.

In this case, c and d are two distinct double curves and this is the inverse of case 1.

3. a and b are part of a single double curve which we call a, and a and b are oriented as follows:

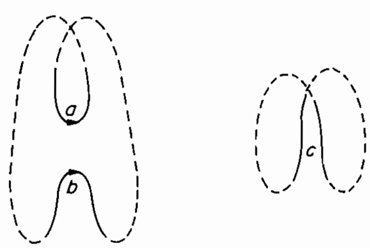


Fig. 19.

Then c and d are part of a single double curve which we call c. We divide a into four arcs: a_1, a_2, A and B, and similarly c as below:

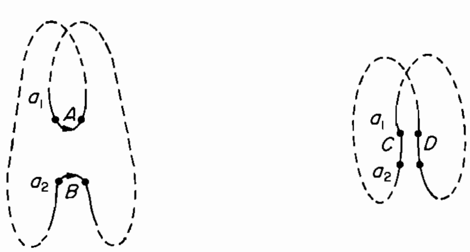


Fig. 20.

Correspondingly we divide a' and b'. Then

$$lk(a,a') = lk(a_1,a_1') + lk(a_2,a_2') + (2k+1)l_a$$

by similar reasoning to case 1, except that here a_1 and a_2 always intersect in an odd number of points. Also

$$lk(c,c') = lk(a_1,a_1')\frac{l_c}{l_a} - lk(a_2,a_2')\frac{l_c}{l_a} + (2j+1)l_c,$$

where j is some integer and $1k(a_2, a_2')\ l_c/l_a$ has a negative sign because the arc reverses orientation. It follows that

$$\frac{m_c}{l_c} = \frac{m_a}{l_a} + 2n \quad \text{for some } n \text{ in } \mathbb{Z}.$$

Note. Clearly $T(f)$ can be similarly defined for S^3. In fact it is possible to define it in any homology sphere or homology $\mathbb{R}^3$. One must then be more careful about the definition of linking and changing a crossing, but the above arguments go through in the new setting. In fact, if we consider only orientable surfaces and 3-manifolds, then an analogous invariant taking values in $\mathbb{Z}_2$ (instead of $\mathbb{Z}_8$) is defined for $\mathbb{Z}_2$-homology spheres or $\mathbb{R}^3$'s.

We are now in a position to solve the problem of which regular homotopy classes in $\mathbb{R}^3$ can be realized by embeddings.

THEOREM 3.8. *Let $f: F \to \mathbb{R}^3$ be a general position immersion of an oriented surface into $\mathbb{R}^3$. Then f is regularly homotopic to an embedding if and only if $T(f) = 0$.*

Proof. Clearly if f is regularly homotopic to an embedding then $T(f) = 0$. Fix an arbitrary embedding $f_0: F \to \mathbb{R}^3$, and suppose $T(f) = 0$. f induces a map $\phi: H_1(F) \to \mathbb{Z}_2$ via the correspondences of Theorem 1.1. Thus f is regularly homotopic to an immersion g which is a standard representative of the homomorphism ϕ. A kink along α_i or β_i alone does not change $T(f)$, but a kink along $\alpha_i + \beta_i$ changes $T(f)$ by 1. Since $T(g) = 0$, g has an even number of kinks along the curves $(\alpha_i \pm \beta_i)$. We can assume that these occur for $i = 1, 2, \ldots, 2k-1, 2k$.

Consider the embedding h obtained by starting with f_0 and doing a Dehn twist along

(1) α_i if $\phi(\beta_i) = 1$ and $\phi(\alpha_i) = 0$.

(2) β_i if $\phi(\alpha_i) = 1$ and $\phi(\beta_i) = 0$.

(3) $\alpha_{2i} - \beta_{2i} + \alpha_{2i+1} - \beta_{2i+1}$ if $1 \leqq i \leqq k$.

We thus get an embedding $f_1: F \to M$ which is the composition of f_0 and a member of the mapping class group of F. This embedding induces a homomorphism $\lambda: H_1(F) \to \mathbb{Z}_2$ which agrees with ϕ, since a frame pushed forward by either immersion twists the same number of times along any given curve (cf. the end of the construction in section 2); thus the embedding is regularly homotopic to f.

We now give a simple proof of a theorem of Banchoff.

THEOREM 3.9. *Let $f: F \looparrowright \mathbb{R}^3$ be a general position immersion. Then the number of triple points of f is equal to the Euler characteristic of F (mod 2).*

Proof. By Corollary 3.4 it suffices to check this for one representative of each regular homotopy class of maps of F into $\mathbb{R}^3$. In case F is orientable, we have seen that every regular homotopy class can be realized without triple points because of Theorem 3.5.

If F is a projective plane, there are two regular homotopy classes of immersions of F into $\mathbb{R}^3$. The left-handed Boy's surface gives one immersion of $\mathbb{R}P^2$ into $\mathbb{R}^3$ and the second can be constructed by adding a kink to this immersion along the single nontrivial curve in $\mathbb{R}P^2$. The resulting immersed $\mathbb{R}P^2$ has an odd number of triple points. It can be constructed as below so as to have two additional triple points for a total of three. In fact, it is regularly homotopic to the right-handed Boy's surface with a single triple point. Thus the theorem holds for each of these.

If F is a Klein bottle, it can be immersed in $\mathbb{R}^3$ with no triple points.

Adding kinks along appropriate curves to realize each regular homotopy class of immersions results in the following three situations:

In each case the number of triple points is even.

If F is an arbitrary nonorientable surface, then it can be constructed from either the projective plane or the Klein bottle by adding handles (connect-summing with tori). Adding

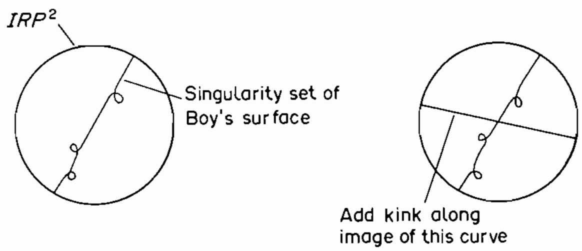


Fig. 21.

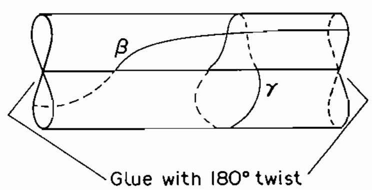


Fig. 22.

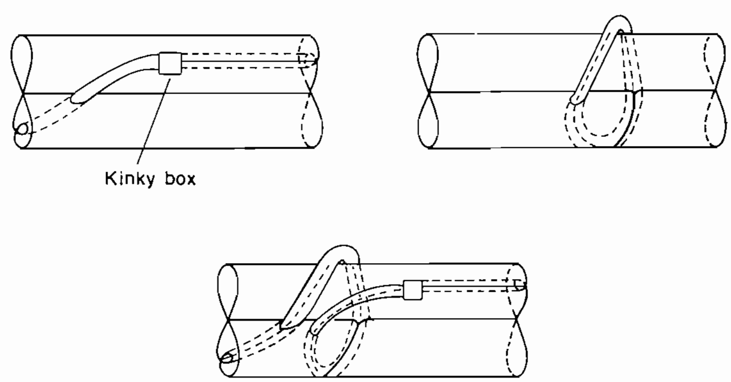


Fig. 23.

kinks along appropriate curves on these handles does not change the number of triple points of the immersion. The result follows.

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