Growth Rate of the Cube Recurrence

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Abstract

In this thesis, we first give a brief overview of various recurrence relations that exhibit rich combinatorial structure. Then we provide a detailed description of the cube recurrence, which is a well-known three-dimensional lattice recurrence relation. We define two types of initial conditions for the cube recurrence, which we refer to as staircase and three-way block conditions. These initial conditions, are easily parameterized by $n \in \mathbb{N}$, and we prove the asymptotic polynomial growth rate of the cube recurrence for each of these types of initial conditions as n tends to ∞.

1 Introduction

There are a host of lattice recurrence relations (indexed by points in the integer lattice $\mathbb{Z}^n$, for some n) that have been studied extensively in combinatorics literature for the past two decades. These recurrences are often formulated in order to better understand phenomena related to various problems in physics. For example, a function on $\mathbb{Z}^3$ satisfies the *octahedron recurrence* if the following relation holds:

$$f_{i-1,j,k-1}\, f_{i,j-1,k} = f_{i,j-1,k-1}\, f_{i-1,j,k} + f_{i-1,j-1,k}\, f_{i,j,k-1}\,.$$

This simple recurrence is also known as the *Hirota bilinear difference equation* and has been shown to generalize many soliton equations [6], where solitons can be thought of as wave pulses that propagate at constant velocity. The octahedron recurrence is also intimately related to Aztec tilings; this is a well-known statistical mechanical ensemble [5] that plays a role in solutions to the square-ice model, a two-dimensional representation of crystal lattices with hydrogen bonds [10]. Another lattice recurrence on $\mathbb{Z}^3$ is given by the relation

$$f_{i,j,k}\, f_{i-1,j-1,k-1} = f_{i-1,j,k}\, f_{i,j-1,k-1} + f_{i,j-1,k}\, f_{i-1,j,k-1} + f_{i,j,k-1}\, f_{i-1,j-1,k}$$

which is known as the *cube recurrence* or *Miwa equation* [5]. The terms in the above equation can be geometrically identified with vertices of a cube, as depicted in Figure 1 below. This recurrence also has connections to classes of nonlinear partial differential equations that describe the physics of certain waves [7]. In addition, the cube recurrence admits a statistical mechanical interpretation in terms of combinatorial objects known as *cube groves*; these are forests defined on an infinite hexagonal grid, whose "entropy" is limited by the particular form of the initial conditions used to setup the cube recurrence (see [1]). We will reference this model in the following section.

More recently, Kenyon and Pemantle [5] analyze the *hexahedron recurrence*, a recurrence relation that is defined on the faces and vertices of the usual cubic tiling of $\mathbb{Z}^3$. This recurrence also lends itself to another statistical mechanics model called taut double-dimer configurations. In [8], Henriques and Speyer define the so-called *multidimensional cube recurrence*, a generalization of the octahedron and cube recurrences that once again admits a combinatorial interpretation.

When the recurrences described above are initialized with formal variables, i.e. set $f_{i,j,k} = x_{i,j,k}$, for sets of initial conditions that take specific prescribed forms, they generate Laurent polynomials (polynomials in which variables can have positive or negative powers) in these formal variables. Fomin and Zelevinsky [9] established this Laurent phenomenon for the octahedron and cube recurrences, and the Laurent property of the hexahedron recurrence is proved in [5]. It can be shown that the monomials in these Laurent polynomials

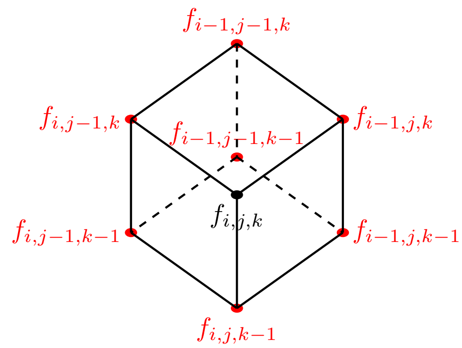

Figure 1: Geometry of the cube recurrence.

are in bijection with the combinatorial configurations of the associated statistical mechanical model; e.g. with a fixed set of initial conditions $\mathcal{I} \subset \mathbb{Z}^3$, the cube recurrence generates Laurent polynomials whose monomial decomposition corresponds to cube groves on $\mathcal{I}$ [1]. Figure 2 provides examples of an Aztec tiling, cube grove, and taut double-dimer configuration. Upon examination of these images, it appears that the randomness of the Aztec tiling and the cube grove is contained within the inscribed circle of the bounding box used for the initial conditions of the recurrence. It turns out that this is indeed the case; in the limit, as the lengths of the bounding box dimensions go to infinity, the entropy (with respect to probability measures defined in [11], [12]) of these combinatorial objects is trapped within this inscribed circle. This is the celebrated Arctic Circle Theorem, which was proven for Aztec tilings by Jockusch et. al [11] and for cube groves by Peterson and Speyer [12].

Note that if we instead initialize these recurrences with numerical values, $f_{i,j,k} = c_{i,j,k}$, then the recurrence will be given by a function $f : \mathbb{Z}^3 \to \mathbb{C}$. In this thesis, we will focus on the cube recurrence with $f_{i,j,k} = 1$ for $(i, j, k) \in \mathcal{I}$, where $\mathcal{I}$ is the set of initial conditions. We consider a fixed point $(i, j, k) \in \mathbb{Z}^3$ and study the value of $f_{i,j,k}$ as the set $\mathcal{I}$ is varied in a controlled fashion; this process will be made clear in the sections to come. Our analysis leads to novel asymptotic results for what we call *staircase* and *three-way block* initial conditions. These configurations will be described in detail in Sections 3 and 4.

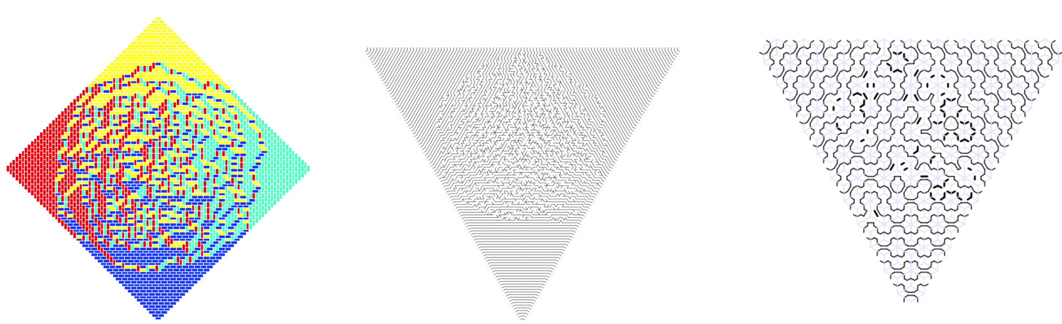

Figure 2: Left: an Aztec tiling. Middle: a cube grove. Right: a double-dimer configuration.

2 The Cube Recurrence

In order to provide a formal definition of the cube recurrence, we first reproduce the mathematical setting established by Carroll and Speyer [1].

Definition 1

The *lower cone* of a point $(i, j, k) \in \mathbb{Z}^3$ is the set

$$C(i, j, k) = \{(i', j', k') \in \mathbb{Z}^3 : i' \leq i,\ j' \leq j,\ k' \leq k\}.$$

Definition 2

Let $\mathcal{L} \subset \mathbb{Z}^3$ be any subset that satisfies the following property: if $(i, j, k) \in \mathcal{L}$, then $C(i, j, k) \subset \mathcal{L}$.

- Define $\mathcal{U} = \mathbb{Z}^3 \setminus \mathcal{L}$.
- Define $\mathcal{K}$ to be the following set:

$$\mathcal{K} = \{(i, j, k) \in \mathcal{U} : 0 < |C(i, j, k) \cap \mathcal{U}| < \infty\}.$$

- The set of *initial conditions* $\mathcal{I}$ with respect to $\mathcal{L}$ is defined as

$$\mathcal{I} = \{(i, j, k) \in \mathcal{L} : (i + 1, j + 1, k + 1) \in \mathcal{U}\}.$$

With these definitions, we see that $\mathcal{L}$ is an order-ideal in $\mathbb{Z}^3$ with respect to the standard lattice ordering, and we can think of the initial conditions $\mathcal{I}$ as the set of points on the surface of $\mathcal{L}$. Intuitively, the set $\mathcal{K}$ consists of points that are "supported" by $\mathcal{I}$ in the following sense: if $(i, j, k) \in \mathcal{K}$, then there exist points of the form $(i', j, k), (i, j', k), (i, j, k') \in \mathcal{I}$.

Now, for any fixed order-ideal $\mathcal{L}$ (with $\mathcal{U}, \mathcal{K}, \mathcal{I}$ defined as above), let $f_{i,j,k}$ be a function that assigns values to points in the set $\mathcal{K} \cup \mathcal{I}$. For $(i, j, k) \in \mathcal{I}$, $f_{i,j,k} = 1$; for $(i, j, k) \in \mathcal{K}$, the function value is defined via the cube recurrence

$$f_{i,j,k} = \frac{f_{i-1,j,k}\, f_{i,j-1,k-1} + f_{i,j-1,k}\, f_{i-1,j,k-1} + f_{i,j,k-1}\, f_{i-1,j-1,k}}{f_{i-1,j-1,k-1}}\ . \tag{1}$$

This function is well-defined, as we now verify.

Lemma 1

Let $\mathcal{L} \subset \mathbb{Z}^3$ be an order-ideal with the corresponding sets $\mathcal{U}, \mathcal{K}, \mathcal{I}$. Then the function f is well-defined on the set $\mathcal{K} \cup \mathcal{I}$.

Proof: It is sufficient to check that f is well-defined on $\mathcal{K}$, since $\mathcal{K} \cap \mathcal{I} = \emptyset$ by examining Definition 2. The claim follows by induction on the coordinate sum of points in $\mathcal{K}$. Consider an arbitrary point $(i, j, k) \in \mathcal{K}$. According to the cube recurrence, $f_{i,j,k}$ depends on the function values for the points in the set

$$A = \{(i - 1, j, k),\ (i, j - 1, k - 1),\ (i, j - 1, k),\ (i - 1, j, k - 1),$$

$$(i, j, k-1),\ (i-1, j-1, k),\ (i-1, j-1, k-1)\} \,.$$

Each of these points $(i', j', k') \in A$ has a coordinate sum less than (i, j, k); i.e. $i' + j' + k' < i + j + k$. In addition, it is easy to see that each of the points in A is either in $\mathcal{I}$ or in $\mathcal{K}$; this follows from the fact that $|C(i, j, k) \cap \mathcal{U}| > 0$. If $(i', j', k') \in \mathcal{I}$, we have $f_{i',j',k'} = 1$. If $(i', j', k') \in \mathcal{K}$, then $f_{i',j',k'}$ has a well-defined value by inductive assumption. We conclude that $f_{i,j,k}$ is well-defined.

If we look at the form of the cube recurrence given by equation (1), we see that the value of the recurrence at a particular point $(i, j, k) \in \mathcal{K}$ only depends on values of the recurrence for points in $C(i, j, k)$. Thus, without loss of generality, we can replace $\mathcal{L}$ with $C(i, j, k) \cap \mathcal{L}$, and update the corresponding sets $\mathcal{U}, \mathcal{K}, \mathcal{I}$ accordingly. We can also translate the lattice by $(i, j, k) \to (0, 0, 0)$, so instead of investigating the value $f_{i,j,k}$ for arbitrary (i, j, k), we can focus our intention on $f_{0,0,0}$. With these simplifications, we have that $\mathcal{L} \subset C(0, 0, 0)$ and $\mathcal{K} = C(0, 0, 0) \setminus \mathcal{L}$. For the rest of this thesis, we will construct order-ideals $\mathcal{L}$ (with corresponding set $\mathcal{K}$) that satisfy these properties.

Furthermore, the cube recurrence value $f_{0,0,0}$ is always a positive integer. One way to see this is to refer to Theorem 1 of Carroll and Speyer [1]; this theorem implies that, with $f_{i,j,k}$ defined as above, $f_{0,0,0}$ counts the number of cube groves on $\mathcal{I}$ (cube groves were described qualitatively in the previous section, and a formal definition is provided in [1]). In this thesis, we consider sequences of initial conditions (I_n), where each initial condition produces a value of the cube recurrence at the origin, $f^{(n)}_{0,0,0}$. It is then possible to analyze the asymptotic growth rate of this sequence of values. For example, Propp [2] found that for what he called *standard initial conditions* obtained from the sets

$$\mathcal{L}_n = \{(i, j, k) \in C(0, 0, 0) \mid i + j + k \leq 1 - n\} \ ,$$

the cube recurrence value at the origin was given by $f^{(n)}_{0,0,0} = 3^{\lfloor n^2/4 \rfloor}$. In this case, the value grows exponentially in the square of the parameter n. $\mathcal{I}_1$ and $\mathcal{I}_4$, the initial conditions corresponding to $\mathcal{L}_1$ and $\mathcal{L}_4$ as defined above, are illustrated in Figure 3. For the families of initial conditions defined in this thesis, we find that the cube recurrence value grows as a polynomial in n.

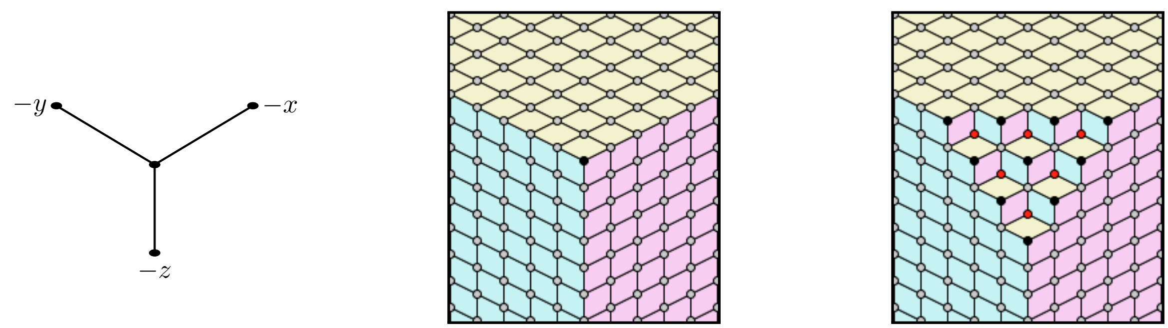

Figure 3: Left: Orientation of the axes in $\mathbb{Z}^3$. Middle: Initial conditions $\mathcal{I}_1$ corresponding to Propp's $\mathcal{L}_1 = C(0, 0, 0)$. Right: Initial conditions $\mathcal{I}_4$ corresponding to Propp's $\mathcal{L}_4$.

3 Staircase Initial Conditions

First, we analyze the growth rate of the cube recurrence on so-called staircase initial conditions. We provide a formal definition for these geometric configurations, but intuitively, such initial conditions are created by pushing a fixed "slider" across the negative orthant of $\mathbb{Z}^3$ to a depth n as illustrated in Figure 4 below.

Definition 3

Staircase initial conditions are a set of initial conditions $\mathcal{I}_n$ that can be obtained in the following fashion:

(1) Pick lattice points on the y, z axes $(0, \hat{j}, 0)$, $(0, 0, \hat{k})$ respectively, with $\hat{j}, \hat{k} \leq -1$.

(2) Select a lattice path P' between the points $(0, \hat{j}, -1)$, $(0, -1, \hat{k})$ such that all edges in the path are either of the form $\{(0, j, k) \to (0, j+1, k)\}$ or $\{(0, j, k) \to (0, j, k-1)\}$. In other words, the path only turns "right" or "down". For later convenience, we define the full path P to be the path P' together with the vertices $\{(0, \hat{j}, 0), (0, 0, \hat{k})\}$ and the incident edges $\{(0, \hat{j}, 0) \to (0, \hat{j}, -1)\}$, $\{(0, -1, \hat{k}) \to (0, 0, \hat{k})\}$.

(3) Let $\mathcal{S}'$ be the set of all vertices $(0, j, k)$ such that the edges $\{(0, j-1, k) \to (0, j, k)\}$, $\{(0, j, k) \to (0, j, k-1)\}$ are in the path P. We then augment $\mathcal{S}'$ by setting $\mathcal{S} = \mathcal{S}' \cup \{(0, \hat{j}, 0), (0, 0, \hat{k}), (-n, 0, 0)\}$, for some $n \geq 0$.

(4) Now we can define

$$\mathcal{L}_n = \bigcup_{(i,j,k) \in \mathcal{S}} C(i, j, k) .$$

Let $\mathcal{U}_n = \mathbb{Z}^3 \setminus \mathcal{L}_n$, $\mathcal{K}_n = C(0, 0, 0) \setminus \mathcal{L}_n$, and $\mathcal{I}_n$ the corresponding set of initial conditions.

Definition 4

A *slider* (T) is constructed by first generating a path P as in Definition 3 and augmenting its vertex set as follows:

$$T = \bigcup_{(0,j,k) \in P} \{(0, j, k') \mid k \leq k' \leq 0\} .$$

Furthermore, let $\|T\|$ denote the area of T; i.e. the number of unit squares given by quadruples of the form $\{(0, j, k), (0, j-1, k), (0, j-1, k-1), (0, j, k-1)\}$ contained in T.

Finally, a *subslider* $T_{p,q}$ of a slider T, where $(0, p, q) \in T$, consists of all points $(0, p', q') \in T$ such that $p' \leq p$, $q' \leq q$, and the corresponding unit square $\{(0, p', q'), (0, p'-1, q'), (0, p'-1, q'-1), (0, p', q'-1)\} \subset T$.

It should be clear from the above two definitions that staircase initial conditions are in one-to-one correspondence with pairs (T, n), where T is a slider and $n \geq 0$. Evidently, given staircase initial conditions, we can construct the pair (T, n) using the path P that was used in Definition 3. Conversely, given a pair (T, n), we can recover the path P used to construct T by computing

$$\hat{j} = \min \{j : (i, j, k) \in T\} ,$$

$$\hat{k} = \min \{k : (i, j, k) \in T\} ,$$

$$Q = \{(0, j, 0) \in T : j > \hat{j}\} \cup \{(0, 0, k) \in T : k > \hat{k}\} ,$$

$$P = T \setminus Q ,$$

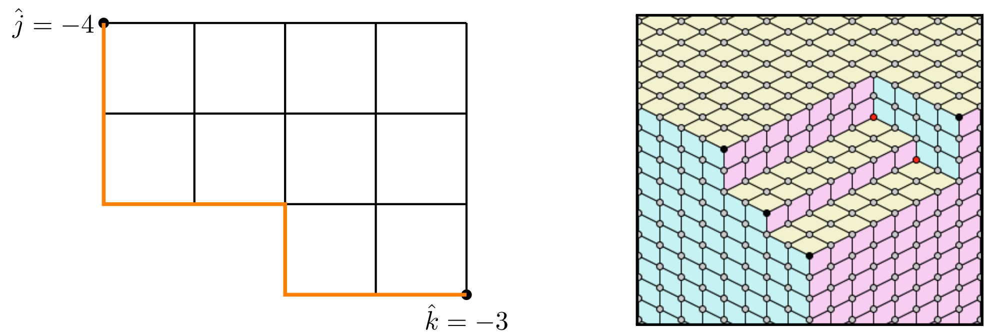

Figure 4: Left: depiction of a slider T with corresponding path P highlighted. Right: staircase initial conditions $\mathcal{I}_n$ for this slider T and depth $n = 7$.

and Definition 3 generates the staircase initial conditions corresponding to P and n. We have now formalized the intuitive notion that a staircase configuration is generated by pushing a fixed slider to some depth n.

Now we will work towards our first main result which characterizes the growth rate of the cube recurrence for a sequence of staircase initial conditions $(\mathcal{I}_n)$ generated by a sequence of pairs $((T, n))$ for a fixed slider T and varying depth n. Our experiments show that the value $f^{(n)}_{0,0,0}$, where the superscript n denotes the depth of the slider T, is given by a polynomial in n. However, for the sake of simplicity, we focus our attention on just calculating the degree of this polynomial. It turns out, that for such a sequence (I_n) with a fixed slider T, this degree is exactly $\|T\|$. Thus, somewhat surprisingly, sliders with the same area produce polynomial formulas for $f^{(n)}_{0,0,0}$ with the same degree, and other geometric properties of the slider simply do not factor into the determination of this growth rate exponent. First, we need to introduce some preliminary notation that will be used throughout the rest of this thesis. Instead of using standard asymptotic notations $\Theta, O, \sim$, we resort to the notation ℓ which is a slightly relaxed version of $\sim$.

Definition 5

Let $f, g : \mathbb{N} \to \mathbb{N}$. We say $f = \ell(g)$ if there is a constant $L > 0$ such that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L\,.$

We maintain that the equality $f = \ell(g)$ stands for set membership $f \in \ell(g)$, as is customary for other asymptotic characterizations. The following lemma states some basic but important properties of ℓ-notation.

Lemma 2

Let $f, g : \mathbb{N} \to \mathbb{N}$.

1) $f = \ell(g)$ if and only if $g = \ell(f)$.

2) $\ell(f)\ell(g) \subset \ell(fg)$.

3) $\frac{\ell(f)}{\ell(g)} \subset \ell\left(\frac{f}{g}\right)$.

4) If $f(n) = \ell(n^t)$ for an integer $t \geq 0$, then $f(n-1) = \ell(n^t)$.

5) If $f(n) - f(n-1) = \ell(n^t)$ for an integer $t \geq 0$, then $f(n) = \ell(n^{t+1})$.

Proof:

1) This follows directly from the symmetry of the definition.

2) Let $F = \ell(f)$ and $G = \ell(g)$ so that $\frac{F}{f} \to k_1$ and $\frac{G}{g} \to k_2$. Then $\frac{FG}{fg} = \frac{F}{f} \cdot \frac{G}{g} \to k_1 k_2$ so $FG = \ell(fg)$ as needed.

3) Using the same setup as in the previous part, $\frac{F/G}{f/g} = \frac{F}{f} \cdot \frac{g}{G} \to \frac{k_1}{k_2}$ which implies that $\frac{F}{G} = \ell(\frac{f}{g})$.

4) The fact that $\frac{f(n)}{n^t} \to k$ implies that $\frac{f(n-1)}{(n-1)^t} \to k$. It follows that $f(n-1) = \ell((n-1)^t)$. Now suppose $F = \ell((n-1)^t)$, i.e. $\frac{F}{(n-1)^t} \to k_1$. Then, using the fact that $\frac{(n-1)^t}{n^t} \to 1$, $\frac{F}{(n-1)^t} \cdot \frac{(n-1)^t}{n^t} \to k_1$ which implies that $F = \ell(n^t)$. Conversely, if $F = \ell(n^t)$, i.e. $\frac{F}{n^t} \to k_2$, then using the fact that $\frac{n^t}{(n-1)^t} \to 1$, we see that $\frac{F}{n^t} \cdot \frac{n^t}{(n-1)^t} \to k_2$ which implies that $F = \ell(n^t)$. We conclude that $\ell((n-1)^t) = \ell(n^t)$, so $f(n-1) = \ell(n^t)$.

5) This is a finite calculus result. From the assumption, $\frac{f(n)-f(n-1)}{n^t} \to k$, so for $\epsilon > 0$, we have that for $n > n_0$
$$\left| \frac{f(n) - f(n-1)}{n^t} - k \right| < \epsilon \,,$$
or equivalently,
$$kn^t - \epsilon < f(n) - f(n-1) < kn^t + \epsilon \,.$$
Summing over n from $n_0 + 1$ to some N, we see that
$$\sum_{n=n_0+1}^{N} (kn^t - \epsilon) < f(N) - f(n_0) < \sum_{n=n_0+1}^{N} (kn^t + \epsilon) \qquad \Longrightarrow \tag{2}$$
$$n_0\epsilon - N\epsilon + p_{t+1}(N) < f(N) - f(n_0) < N\epsilon - n_0\epsilon + p_{t+1}(N) \,, \tag{3}$$
where $p_{t+1}(N) = \sum_{n=n_0+1}^{N} kn^t$ is a polynomial in N of degree $t+1$. Clearly $p_{t+1}(N) = \ell(N^{t+1})$, so let $c = \lim_{N\to\infty} \frac{p_{t+1}(N)}{N^{t+1}}$. Now by dividing line (3) above by N^{t+1}, we obtain
$$\frac{n_0\epsilon}{N^{t+1}} - \frac{\epsilon}{N^t} + \frac{p_{t+1}(N)}{N^{t+1}} + \frac{f(n_0)}{N^{t+1}} < \frac{f(N)}{N^{t+1}} < \frac{\epsilon}{N^t} - \frac{n_0\epsilon}{N^{t+1}} + \frac{p_{t+1}(N)}{N^{t+1}} + \frac{f(n_0)}{N^{t+1}} \,.$$
It is now evident that in the limit as N goes to ∞, $\frac{f(N)}{N^{t+1}} \to c$, so $f(N) = \ell(N^{t+1})$ as needed.

We are now ready to derive an asymptotic result for staircase initial conditions.

Theorem 1

Let T be a slider that characterizes a sequence of staircase initial conditions $(\mathcal{I}_n)$ where n varies over $\mathbb{N}$. Furthermore, we initialize each $\mathcal{I}_n$ with a function value of 1 on each vertex; i.e. for each $(i,j,k) \in \mathcal{I}_n$, we set $f^{(n)}_{i,j,k} = 1$. Then the cube recurrence value at the origin satisfies $f^{(n)}_{0,0,0} = \ell(n^{\|T\|})$.

Proof: Let $\mathcal{K}_n$ be defined with respect to $\mathcal{I}_n$ as in Definition 3. First, we assign a function $g_{j,k} : \mathbb{N} \to \mathbb{R}$ to each point $(0,j,k) \in T$, with the rule $g_{j,k}(n) = f^{(n)}_{0,j,k}$. Note that for points $(0,j,k) \in T \cap \mathcal{I}_n$, $g_{j,k}$ will be a constant function taking the value 1, since f assigns a value of 1 to points on the initial conditions. The cube recurrence gives us a relation amongst these new functions. For the sake of notational sanity, let $W = g_{j-1,k}$, $X = g_{j-1,k-1}$, $Y = g_{j,k-1}$, $Z = g_{j,k}$ when the indices j,k are fixed; furthermore, suppose that $W = \ell(n^w)$, $X = \ell(n^x)$, $Y = \ell(n^y)$ with $w,x,y \geq 0$ and $w + y - 2x \geq 0$. Then for $(0,j,k) \in T \cap \mathcal{K}_n$, we have

$$Z(n) = \frac{Z(n-1)X(n) + W(n)Y(n-1) + Y(n)W(n-1)}{X(n-1)} \implies \tag{4}$$

$$Z(n)X(n-1) - Z(n-1)X(n) = W(n)Y(n-1) + Y(n)W(n-1) \implies \tag{5}$$

$$\frac{Z(n)}{X(n)} - \frac{Z(n-1)}{X(n-1)} = \frac{W(n)Y(n-1) + Y(n)W(n-1)}{X(n)X(n-1)} = \ell(n^{w+y-2x}) \implies \tag{6}$$

$$\frac{Z(n)}{X(n)} = \ell(n^{w+y-2x+1}) \implies \tag{7}$$

$$Z(n) = X(n)\ell(n^{w+y-2x+1}) = \ell(n^{w+y-x+1}) \tag{8}$$

where step (6) follows from parts 2), 3), and 4) of Lemma 2, step (7) is an application of part 5), and step (8) again uses part 2).

Thus, if we associate the asymptotic exponent $z = w + y - x + 1$ obtained via this calculation to the function Z, say $e(Z) = w + y - x + 1$, we obtain a two-dimensional recurrence on $T \cap \mathcal{K}_n$ given by

$$e(g_{j,k}) = e(g_{j-1,k}) + e(g_{j,k-1}) - e(g_{j-1,k-1}) + 1\,, \tag{9}$$

and $e(g_{j,k}) = 0$ for $(0,j,k) \in T \cap \mathcal{I}_n$ since we assigned constant functions $g_{j,k}$ to these vertices. Note that step (7) of the computation above relied on the fact that the asymptotic exponent was nonnegative. We will now prove, by induction on the area of the subsliders $T_{j,k} \subset T$ (as in Definition 4), that $e(g_{j-1,k}) + e(g_{j,k-1}) - 2 \cdot e(g_{j-1,k-1}) \geq 0$ and $e(g_{j,k}) = \|T_{j,k}\|$ for all $g_{j,k} \in T \cap \mathcal{K}_n$. The main idea behind the induction is illustrated in Figure 5 below.

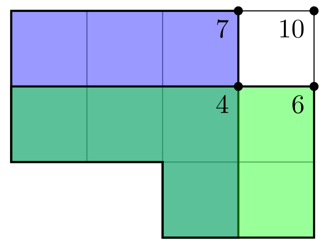

Figure 5: The slider T from Figure 4 with highlighted subsliders; $7 + 6 - 4 + 1 = 10$.

If $T_{j,k}$ is empty, we say $\|T_{j,k}\| = 0$ by convention. Note that if T contains only one square, then three of the four vertices must lie in $\mathcal{I}_n$. It follows that $e(g_{-1,0}) + e(g_{0,-1}) - 2 \cdot e(g_{-1,-1}) = 0 + 0 - 0 \geq 0$ and $e(g_{0,0}) = e(g_{-1,0}) + e(g_{0,-1}) - e(g_{-1,-1}) + 1 = 0 + 0 - 0 + 1 = 1 = \|T_{0,0}\|$. Now consider a general slider T, and suppose the claim holds for all $T_{j',k'}$ with $\|T_{j',k'}\| < \|T_{j,k}\|$. We have that

$$e(g_{j-1,k}) + e(g_{j,k-1}) - 2 \cdot e(g_{j-1,k-1}) = \|T_{j-1,k}\| + \|T_{j,k-1}\| - 2\,\|T_{j-1,k-1}\| \geq 0$$

since $T_{j-1,k-1} \subset T_{j-1,k}$ and $T_{j-1,k-1} \subset T_{j,k-1}$. Now, we are justified in applying the recurrence (10) and we see that

$$\begin{aligned} e(g_{j,k}) &= e(g_{j-1,k}) + e(g_{j,k-1}) - e(g_{j-1,k-1}) + 1 \\ &= \|T_{j-1,k}\| + \|T_{j,k-1}\| - \|T_{j-1,k-1}\| + 1 \\ &= \|T_{j,k}\| \ . \end{aligned}$$

where we have used the fact that all the subsliders $T_{j-1,k}$, $T_{j,k-1}$, $T_{j-1,k-1}$ have strictly smaller area than $T_{j,k}$ (all three omit the square $\{(0,j,k), (0,j-1,k), (0,j-1,k-1), (0,j,k-1)\} \in T_{j,k}$). This completes the induction. In particular, we have that $e(g_{0,0}) = \|T_{0,0}\| = \|T\|$, which means that $g_{0,0}(n) = \ell(n^{\|T\|})$. Since $g_{0,0}(n) = f^{(n)}_{0,0,0}$ for the staircase initial conditions $\mathcal{I}_n$, we have proven the desired result.

4 Three-way Block Initial Conditions

The previous section and Theorem 1 addressed the case in which we push a slider across a single axis in $\mathbb{Z}^3$. It then seems natural to investigate what happens when a slider is pushed simultaneously across all three axes. Intuitively, as the sliders are pushed farther and farther along the axes, they effectively cease to interact with each other, and the growth rate of the cube recurrence value should act in accordance with Theorem 1. However, there will be a region close to the origin where the sliders overlap as they are pushed in different directions. If we are allowed to select an arbitrary slider for each axis, the geometry of the region of overlap can become incredibly complicated. To simplify matters, we consider the special case in which the three selected sliders are rectangles that "fit together" to form the sides of a rectangular prism; an illustration of this configuration is provided in Figure 6 below. This particular case admits a neat inductive proof for the verification of the growth rate exponent formula that we discovered. We start by providing a formal definition for the specific form of these initial conditions, and then we work towards the statement of the next main result.

Definition 6

Three-way block initial conditions $\mathcal{I}_n$ are parametrized by a point and a depth, $((x,y,z),n)$ for integers $x,y,z,n \geq 0$. We then set

$$\mathcal{L}_n = C(-x-n,0,0) \cup C(0,-y-n,0) \cup C(0,0,-z-n) \\ \cup C(-x,-y,0) \cup C(-x,0,-z) \cup C(0,-y,-z)$$

with $\mathcal{U}_n$ and $\mathcal{I}_n$ following suit as in Definition 1.

We would like to analyze the asymptotic growth rate of the cube recurrence value at the origin as n tends to infinity in the above definition. Note that for points $\mathbf{p} = (-x,b,c)$ in the set $P = \{(-x,j,k) \in \mathbb{Z}^3 : -y \leq j \leq 0, -z \leq k \leq 0\}$, $f_{\mathbf{p}} = \ell(n^{(y+b)(z+c)})$ by applying Theorem 1 to the (rectangular) sliders $T_{\mathbf{p}} = \{(-x,j,k) \in \mathbb{Z}^3 : -y \leq j' \leq b, -z \leq k' \leq c\}$ which each have area $\|T_{\mathbf{p}}\| = (y+b)(z+c)$; note that we are implicitly invoking the translation argument presented in Section 1 in order to treat these sets as sliders. Similarly, $f_{\mathbf{q}} = \ell(n^{(x+a)(z+c)})$ for each point $\mathbf{q} = (a,-y,c) \in Q$ where $Q = \{(i,-y,k) \in \mathbb{Z}^3 : -x \leq i \leq 0, -z \leq k \leq 0\}$, and $f_{\mathbf{r}} = \ell(n^{(x+a)(y+b)})$ for each point $\mathbf{r} = (a,b,-z) \in R$ where $R = \{(i,j,-z) \in \mathbb{Z}^3 : -x \leq i \leq 0, -y \leq j \leq 0\}$. We assign this growth rate exponent to each point in the sets P,Q,R as illustrated in Figure 6.

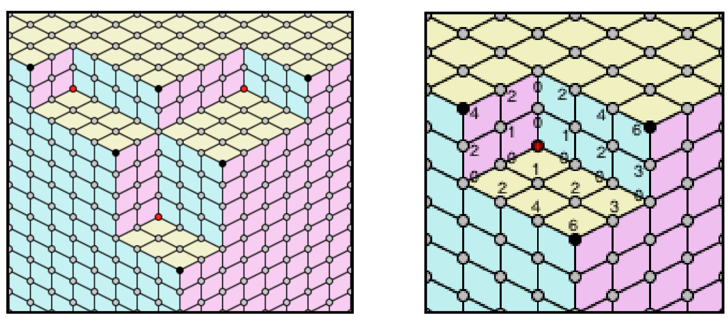

Figure 6: Left: three-way block initial conditions $\mathcal{I}_n$ parametrized by $((x,y,z),n) = ((2,3,2),4)$. Right: the corresponding sets P,Q,R for $\mathcal{I}_n$, decorated with area values; P is the blue rectangle, Q is the red rectangle, and R is the yellow rectangle.

Now, the question is, how do we find the growth rate exponent at the origin, $(0,0,0)$, given the exponents associated to the points in the sets P,Q,R? The answer follows from the cube recurrence, given by equation 1 in Section 2. To make our lives easier, we apply the translation $(i,j,k) \mapsto (i+x, j+y, k+z)$ to the set of vertices in the overlap region of consideration, $S' = \{(i,j,k) \in \mathbb{Z}^3 : -x \leq i \leq 0, -y \leq j \leq 0, -z \leq k \leq 0\}$, to obtain a shifted set of coordinates $S = \{(x',y',z') \in \mathbb{Z}^3 : 0 \leq x' \leq x, 0 \leq y' \leq y, 0 \leq z' \leq z\}$. For some fixed $(x',y',z') \in S$, suppose $f_{u,v,w} = \ell(n^{t(u,v,w)})$ when $u+v+w < x'+y'+z'$ (and $t : \mathbb{Z}^3 \to \mathbb{N}$ is some function). Then we see that

$$f_{x',y',z'} = \frac{f_{x'-1,y',z'}f_{x',y'-1,z'-1} + f_{x',y'-1,z'}f_{x'-1,y',z'-1} + f_{x',y',z'-1}f_{x'-1,y'-1,z'}}{f_{x'-1,y'-1,z'-1}}$$

$$= \frac{\ell(n^{t(x'-1,y',z')})\ell(n^{t(x',y'-1,z'-1)}) + \ell(n^{t(x',y'-1,z')})\ell(n^{t(x'-1,y',z'-1)}) + \ell(n^{t(x',y',z'-1)})\ell(n^{t(x'-1,y'-1,z')})}{\ell(n^{t(x'-1,y'-1,z'-1)})}$$

$$= \ell(n^{\max\{t(x'-1,y',z')+t(x',y'-1,z'-1),t(x',y'-1,z')+t(x'-1,y',z'-1),t(x',y',z'-1)+t(x'-1,y'-1,z')\}-t(x'-1,y'-1,z'-1)}).$$

Thus, $f_{x',y',z'}$ also grows at a polynomial rate, and the growth rate exponents in the set S satisfy a new recurrence relation that is a tropicalization of the original cube recurrence. For clarity, this *tropical recurrence* on the set S is given by

$$\begin{aligned} t(x',y',z') = \max\{&t(x'-1,y',z') + t(x',y'-1,z'-1) - t(x'-1,y'-1,z'-1), \\ &t(x',y'-1,z') + t(x'-1,y',z'-1) - t(x'-1,y'-1,z'-1), \\ &t(x',y',z'-1) + t(x'-1,y'-1,z') - t(x'-1,y'-1,z'-1)\} \end{aligned}$$

along with the boundary conditions $t(0,y',z') = y'z'$, $t(x',0,z') = x'z'$, and $t(0,y',z') = y'z'$ as we calculated previously for the sets P, Q, R above (now with the appropriate shifted coordinates on S). Therefore, we can find the growth rate exponent at the origin (which corresponds to the point (x,y,z) in the set S) by applying the tropical recurrence on S to obtain a value for $t(x,y,z)$. The following theorem solves this recurrence by providing a closed form solution for $t(x,y,z)$.

Theorem 2

Let (x,y,z), for integers $x,y,z \geq 0$, characterize a sequence of three-way block initial conditions $(\mathcal{I}_n)$, where n varies over $\mathbb{N}$. Furthermore, we initialize each $\mathcal{I}_n$ with a function value of 1 on each vertex; i.e. for each $(i,j,k) \in \mathcal{I}_n$, we set $f^{(n)}_{i,j,k} = 1$. Then the cube recurrence value at the origin satisfies $f_{0,0,0} = \ell(n^{b(x,y,z)})$, where

$$b(x,y,z) = \begin{cases} xy + yz - xz\,, & x+z \leq y \\ xz + yz - xy\,, & x+y \leq z \\ xy + xz - yz\,, & y+z \leq x \\ \left\lfloor \frac{x^2+y^2+z^2}{2} \right\rfloor, & \text{otherwise.} \end{cases}$$

Proof: First, we note that the function b is well-defined. While it is possible for (x,y,z) to fall into multiple of the first three pieces of the function definition, the value of $b(x,y,z)$ will coincide on these pieces. For example, if both $x+z \leq y$ and $x+y \leq z$, then $x = 0$, $y = z$ (using the fact that $x,y,z \geq 0$), and it is easy to see that the value of $b(x,y,z)$ on these pieces will be the same. Similar reasoning applies if both $x+y \leq z$ and $y+z \leq x$, or if both $x+z \leq y$ and $y+z \leq x$.

To prove the theorem, we need to verify that the function b satisfies the tropical recurrence on the set S as defined above. First, we note that $b(0,y,z) = yz$, $b(x,0,z) = xz$, and $b(x,y,0) = xy$ which agrees with the base cases of the recurrence relation, so we need only be concerned with the case $x,y,z > 0$ under the inductive assumption that the formula b holds for all $(x',y',z') \in S$ with $x'+y'+z' < x+y+z$.

First, we analyze what happens when (x,y,z) falls into one of the first three pieces of the definition of b. Since the function is symmetric, we can assume without loss of generality that (x,y,z) falls into the first piece; i.e. $x+z \leq y$ so that $b(x,y,z) = xy + yz - xz$. As a sub-case, we first restrict our consideration to $x+z \leq y-1$. Then all terms in the tropical recurrence will fall under the first branch in the definition of b. We have

$$\begin{aligned} b(x,y,z) \stackrel{?}{=} \max\{&b(x-1,y,z) + b(x,y-1,z-1) - b(x-1,y-1,z-1), \\ &b(x,y-1,z) + b(x-1,y,z-1) - b(x-1,y-1,z-1), \\ &b(x,y,z-1) + b(x-1,y-1,z) - b(x-1,y-1,z-1)\} \end{aligned}$$

$$
\begin{aligned}
&= \max\{(x-1)y + yz - (x-1)z + x(y-1) + (y-1)(z-1) - x(z-1), \\
&\qquad x(y-1) + (y-1)z - xz + (x-1)y + y(z-1) - (x-1)(z-1), \\
&\qquad xy + y(z-1) - x(z-1) + (x-1)(y-1) + (y-1)z - (x-1)z\} \\
&\qquad - ((x-1)(y-1) + (y-1)(z-1) - (x-1)(z-1)) \\
&= \max\{2xy - 2xz + 2yz - 2y + 1, \\
&\qquad 2xy - 2xz + 2yz - 2y - 1, \\
&\qquad 2xy - 2xz + 2yz - 2y + 1\} \\
&\qquad - (xy - xz + yz - 2y + 1) \\
&= \qquad xy + yz - xz
\end{aligned}
$$

and the equality is verified. In the other sub-case, $x+z=y$ and the $b(x, y-1, z)$ term in the recurrence will now require the fourth piece of the definition of b.

$$
\begin{aligned}
b(x,y,z) &\stackrel{?}{=} \max\{(x-1)(x+z) + (x+z)z - (x-1)z + x(x+z-1) + (x+z-1)(z-1) - x(z-1), \\
&\qquad \left\lfloor \frac{x^2 + (x+z-1)^2 + z^2}{2} \right\rfloor + (x-1)(x+z) + (x+z)(z-1) - (x-1)(z-1), \\
&\qquad x(x+z) + (x+z)(z-1) - x(z-1) + (x-1)(x+z-1) + (x+z-1)z - (x-1)z\} \\
&\qquad - ((x-1)(x+z-1) + (x+z-1)(z-1) - (x-1)(z-1)) \\
&= \max\{2x^2 + 2xz - 2x + 2z^2 - 2z + 1, \\
&\qquad 2x^2 + 2xz - 2x + 2z^2 - 2z - 1, \\
&\qquad 2x^2 + 2xz - 2x + 2z^2 - 2z + 1\} \\
&\qquad - ((x-1)(x+z-1) + (x+z-1)(z-1) - (x-1)(z-1)) \\
&= \qquad x^2 + xz + z^2 \\
&= \qquad xy + yz - xz
\end{aligned}
$$

and again the equality is verified.

Now we analyze what happens when (x, y, z) falls into the fourth piece of the definition of b. In this case, $x+z > y$, $x+y > z$, and $y+z > x$ (i.e. the lengths form a triangle) and $b(x,y,z) = \lfloor \frac{x^2+y^2+z^2}{2} \rfloor$. As the first sub-case, we consider the situation $x+z > y+1$, $x+y > z+1$, $y+z > x+1$. Then all terms in the tropical recurrence will fall under the fourth piece in the definition of b (noting the simple fact that if we have equality in the condition for any of the first three pieces of b, the fourth piece will yield the same value). Assuming that $x^2+y^2+z^2$ is even, we have

$$
\begin{aligned}
b(x,y,z) \stackrel{?}{=} \max\Bigg\{ &\left\lfloor \frac{(x-1)^2 + y^2 + z^2}{2} \right\rfloor + \left\lfloor \frac{x^2 + (y-1)^2 + (z-1)^2}{2} \right\rfloor, \\
&\left\lfloor \frac{x^2 + (y-1)^2 + z^2}{2} \right\rfloor + \left\lfloor \frac{(x-1)^2 + y^2 + (z-1)^2}{2} \right\rfloor,
\end{aligned}
$$

$$\left\lfloor \frac{x^2+y^2+(z-1)^2}{2} \right\rfloor + \left\lfloor \frac{(x-1)^2+(y-1)^2+z^2}{2} \right\rfloor \Bigg\}$$
$$- \left\lfloor \frac{(x-1)^2+(y-1)^2+(z-1)^2}{2} \right\rfloor$$
$$= \max \Bigg\{ \frac{x^2+y^2+z^2}{2} - x + \frac{x^2+y^2+z^2}{2} - y - z + 1,$$
$$\frac{x^2+y^2+z^2}{2} - y + \frac{x^2+y^2+z^2}{2} - x - z + 1,$$
$$\frac{x^2+y^2+z^2}{2} - z + \frac{x^2+y^2+z^2}{2} - x - y + 1 \Bigg\}$$
$$- \left\lfloor \frac{(x-1)^2+(y-1)^2+(z-1)^2}{2} \right\rfloor$$
$$= \quad x^2+y^2+z^2-x-y-z+1 - \left(\frac{x^2+y^2+z^2}{2} - x - y - z + 1 \right)$$
$$= \quad \left\lfloor \frac{x^2+y^2+z^2}{2} \right\rfloor$$

and the equality is verified. When $x^2+y^2+z^2$ is odd, we obtain the same result:

$$b(x,y,z) \stackrel{?}{=} \max \Bigg\{ \frac{x^2+y^2+z^2+1}{2} - x + \frac{x^2+y^2+z^2+1}{2} - y - z,$$
$$\frac{x^2+y^2+z^2+1}{2} - y + \frac{x^2+y^2+z^2+1}{2} - x - z,$$
$$\frac{x^2+y^2+z^2+1}{2} - z + \frac{x^2+y^2+z^2+1}{2} - x - y \Bigg\}$$
$$- \left\lfloor \frac{(x-1)^2+(y-1)^2+(z-1)^2}{2} \right\rfloor$$
$$= \quad x^2+y^2+z^2-x-y-z+1 - \left(\frac{x^2+y^2+z^2+1}{2} - x - y - z + 1 \right)$$
$$= \quad \left\lfloor \frac{x^2+y^2+z^2}{2} \right\rfloor .$$

The next three sub-cases address the situation when one or more of the following conditions hold: $\{x+z=y+1,\ x+y=z+1,\ y+z=x+1\}$. This would technically yield six more sub-cases, but since b is symmetric, we can save ourselves some work. The second sub-case handles the situation when one equality above holds; without loss of generality $x+z=y+1$ (we still have $x+y>z+1$ and $y+z>x+1$ from before). This amounts to plugging in $y=x+z-1$ into the tropical recurrence, and using the first piece of the definition of b to compute $b(x-1,y,z-1)$ instead of the fourth:

$$b(x,y,z) \stackrel{?}{=} \max \Bigg\{ \left\lfloor \frac{(x-1)^2+(x+z-1)^2+z^2}{2} \right\rfloor + \left\lfloor \frac{x^2+(x+z-2)^2+(z-1)^2}{2} \right\rfloor,$$

$$\left\lfloor \frac{x^2+(x+z-2)^2+z^2}{2} \right\rfloor + (x-1)(x+z-1)+(x+z-1)(z-1)-(x-1)(z-1),$$

$$\left.\left\lfloor \frac{x^2+(x+z-1)^2+(z-1)^2}{2} \right\rfloor + \left\lfloor \frac{(x-1)^2+(x+z-2)^2+z^2}{2} \right\rfloor \right\}$$

$$- \left\lfloor \frac{(x-1)^2+(x+z-2)^2+(z-1)^2}{2} \right\rfloor$$

$$\begin{aligned}
&= 2x^2+2xz-4x+2z^2-4z+3-(x^2+xz-3x+z^2-3z+3) \\
&= x^2+xz-x+z^2-z \\
&= \left\lfloor \frac{x^2+(x+z-1)^2+z^2}{2} \right\rfloor \\
&= \left\lfloor \frac{x^2+y^2+z^2}{2} \right\rfloor .
\end{aligned}$$

The third sub-case handles the situation when two equality conditions hold; without loss of generality, $x+z=y+1$ and $x+y=z+1$ (we still maintain $y+z>x+1$). Note that these two equalities imply $x=1$, and $y=z$, so we can substitute these into the tropical recurrence. Also, we must use the first piece of b to compute $b(x-1,y,z-1)$ and the second piece to compute $b(x-1,y-1,z)$.

$$b(x,y,z) \stackrel{?}{=} \max\left\{ \left\lfloor \frac{(x-1)^2+y^2+z^2}{2} \right\rfloor + \left\lfloor \frac{x^2+(y-1)^2+(z-1)^2}{2} \right\rfloor, \right.$$

$$\left\lfloor \frac{x^2+(y-1)^2+z^2}{2} \right\rfloor + (x-1)y+y(z-1)-(x-1)(z-1),$$

$$\left.\left\lfloor \frac{x^2+y^2+(z-1)^2}{2} \right\rfloor + (x-1)z+(y-1)z-(x-1)y \right\}$$

$$- \left\lfloor \frac{(x-1)^2+(y-1)^2+(z-1)^2}{2} \right\rfloor$$

$$\begin{aligned}
&= \max\{\ y^2+(y-1)^2,\ y^2-y+1+y^2-y,\ y^2-y+1+y^2-y\} - (y-1)^2 \\
&= y^2 \\
&= \left\lfloor \frac{x^2+y^2+z^2}{2} \right\rfloor .
\end{aligned}$$

The fourth sub-case handles the situation when all three equality conditions hold; i.e. $x+z=y+1$, $x+y=z+1$, and $y+z=x+1$. These conditions imply $x=y=z=1$, so we can plug these values into the maximization expression in the tropical recurrence. We must use the first piece of b to compute $b(x-1,y,z-1)$, the second piece to compute $b(x-1,y-1,z)$, and the third piece to compute $b(x,y-1,z-1)$.

$$b(x,y,z) \stackrel{?}{=} \max\left\{ \left\lfloor \frac{(x-1)^2+y^2+z^2}{2} \right\rfloor + x(y-1)+x(z-1)-(y-1)(z-1), \right.$$

$$
\begin{aligned}
& \left\lfloor \frac{x^2 + (y-1)^2 + z^2}{2} \right\rfloor + (x-1)y + y(z-1) - (x-1)(z-1), \\
& \left.\left\lfloor \frac{x^2 + y^2 + (z-1)^2}{2} \right\rfloor + (x-1)z + (y-1)z - (x-1)y \right\} \\
& - \left\lfloor \frac{(x-1)^2 + (y-1)^2 + (z-1)^2}{2} \right\rfloor \\
= \quad & 1 \\
= \quad & \left\lfloor \frac{x^2 + y^2 + z^2}{2} \right\rfloor .
\end{aligned}
$$

This completes the proof of Theorem 2.

5 Conclusion

In this thesis, we derived the growth rate exponent, or polynomial degree, for values taken by the cube recurrence on two specific types of initial conditions: staircase and three-way block configurations. It would be of general interest to characterize the growth rate of the cube recurrence for other families of initial conditions. For the two types of initial conditions that we investigated, it turned out that the cube recurrence values grew at a polynomial rate in the depth parameter n, while the cube recurrence values for Propp's standard initial conditions (described in Section 2) grew exponentially fast in the depth n. We would like to understand the cause of this phenomenon. One potentially relevant observation is that staircase and three-way block initial conditions remove volume from $C(0, 0, 0)$ at a linear rate in the depth n – the sliders "displace" this volume as they are pushed along the axes of $\mathbb{Z}^3$; on the other hand, Propp's standard initial conditions remove volume from $C(0, 0, 0)$ at a cubic rate, since for each n, they remove a regular tetrahedron of side length n. It might be the case that the transition from polynomial to exponential growth in cube recurrence values is correlated with the rate of volume removal associated with the form of the initial conditions.

As mentioned at the start of the thesis, there are many other recurrences that share similar combinatorial properties to the cube recurrence. In particular, this implies that they produce integer values when initialized in an appropriate fashion. Examples include the octahedron and hexahedron recurrences [4, 5]. Future research could establish asymptotic growth rate results for specific initial conditions for these alternate recurrences. Ideally, we would be able to find criterion to distinguish whether the growth rate of the recurrence values for a particular sequence of initial conditions will be polynomial or exponential, just as in the case of the cube recurrence.

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References

[1] G. Carroll, D. Speyer. "The Cube Recurrence", *Electronic Journal of Combinatorics 11* (2004), research paper 74.

[2] J. Propp. "The Many Faces of Alternating-Sign Matrices", *Discrete Mathematics and Theoretical Computer Science Proceedings* AA (DM-CCG) (2001), 43-58.

[3] R.L. Graham, D.E. Knuth, O. Patashnik (1994). *Concrete Mathematics.* New York: Addison-Wesley Publishing Company.

[4] D. Speyer. "Perfect Matchings and the Octahedron Recurrence", *Journal of Algebraic Combinatorics* (2007), Volume 25 Issue 3, 309 - 348.

[5] R. Kenyon, R. Pemantle. "Double Dimers, the Ising Model, and the Hexahedron Recurrence", *Journal of Combinatorial Theory, Series A* (2016), Volume 137 Issue C, 27-63.

[6] A.V. Zabrodin. "A Survey of Hirota's Difference Equations", *Theoretical and Mathematical Physics* (1997), 113:2, 1347 - 1392.

[7] B. Cao. "Solutions of Jimbo-Miwa Equation and Konopelchenko-Dubrovsky Equations", *Acta Applicandae Mathematicae* (2010), Volume 112, 181 - 203.

[8] A. Henriques, D. Speyer. "The Multidimensional Cube Recurrence", *Advances in Mathematics* (2010), Volume 223 Issue 3, 1107 - 1136.

[9] S. Fomin, A. Zelevinsky. "The Laurent Phenomenon", *Advances in Applied Mathematics* (2002), Volume 28 Issue 2, 119-144.

[10] N. Elkies, G. Kuperberg, M. Larsen, J. Propp. "Alternating Sign Matrices and Domino Tilings", *Journal of Algebraic Combinatorics* (1992), Volume 1 Issue 2, 111 - 132.

[11] W. Jockusch, J. Propp, and P. Shor. "Random Domino Tilings and the Arctic Circle Theorem", *arXiv:math/9801068* (1995).

[12] T. Petersen, D. Speyer. "An Arctic Circle Theorem for Groves", *Journal of Combinatorial Theory, Series A* (2005), Volume 11 Issue 1, 137 - 164.