Optimal Upward Planarity Testing of Single-Source Digraphs

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Abstract

A digraph is upward planar if it has a planar drawing such that all the edges are monotone with respect to the vertical direction. Testing upward planarity and constructing upward planar drawings is important for displaying hierarchical network structures, which frequently arise in software engineering, project management, and visual languages. In this paper we investigate upward planarity testing of single-source digraphs: we provide a new combinatorial characterization of upward planarity, and give an optimal algorithm for upward planarity testing. Our algorithm tests whether a single-source digraph with n vertices is upward planar in $O(n)$ sequential time, and in $O(\log n)$ time on a CRCW PRAM with $n \log\log n / \log n$ processors, using $O(n)$ space. The algorithm also constructs an upward planar drawing if the test is successful. The previous best result is an $O(n^2)$-time algorithm by Hutton and Lubiw. No efficient parallel algorithms for upward planarity testing were previously known.

Key Words Graph drawing, planar graph, upward drawing, triconnected components, parallel algorithm.

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1 Introduction

The upward planarity of digraphs is a fundamental issue in the area of graph drawing and has been extensively investigated. A digraph is upward planar if it has a planar upward drawing, i.e. a planar drawing such that all the edges are monotone with respect to the vertical direction (see Fig. 1.a). Planarity and acyclicity are necessary but not sufficient conditions for upward planarity, as shown in Fig. 1.b.

Figure 1: Examples of planar acyclic digraphs: (a) upward planar; and (b) not upward planar.

Testing upward planarity and constructing upward planar drawings is important for displaying hierarchical network structures, which frequently arise in a wide variety of areas. Key domains of application include software engineering, project management, and visual languages. Especially significant in a number of applications are single-source digraphs, such as subroutine-call graphs, is-a hierarchies, and organization charts. Also, upward planarity of single-source digraphs has deep combinatorial implications in the theory of ordered sets. Namely, the orders defined by the transitive closure of upward planar single-source digraphs have bounded dimension [34], so that they can be compactly represented.

A survey on algorithms for planarity testing and graph drawing can be found in [7]. Previous work on upward planarity is as follows.

Combinatorial results on upward planarity for covering digraphs of lattices were first given in [22, 26]. Further results on the interplay between upward planarity and ordered sets are surveyed by Rival [30]. Lempel, Even, and Cederbaum [23] relate the planarity of biconnected undirected graphs to the upward planarity of *st*-digraphs. A combinatorial characterization of upward planar digraphs is provided in [21, 9]: namely, a digraph is upward planar if and only if it is a subgraph of a planar *st-digraph*.

Di Battista, Tamassia, and Tollis [9, 12] give algorithms for constructing upward planar drawings of *st*-digraphs, and investigate area bounds and symmetry display. Tamassia and Vitter [32] show that the above drawing algorithms can be efficiently parallelized. Upward planar drawings of trees and series-parallel digraphs are studied in [29, 31, 6, 13, 15] and [1, 2], respectively.

In [8] it is shown that for the special case of bipartite digraphs, upward planarity is equivalent to planarity. In [3, 4] a polynomial time algorithm is given for testing the upward planarity of digraphs with a prescribed embedding. Thomassen [33] characterizes the upward planarity of single-source digraphs in terms of forbidden circuits. Hutton and Lubiw [19] combine Thomassen's characterization with a decomposition scheme to test the upward planarity of an n-vertex single-source digraph in $O(n^2)$ time. Very recently, Papakostas [25] has given a polynomial-time algorithm for upward planarity testing of outerplanar digraphs, and Garg and Tamassia [16] have shown that upward planarity testing is NP-complete for general digraphs.

In this paper we investigate upward planarity testing of single-source digraphs. Our main results are summarized as follows:

- We provide a new combinatorial characterizations of upward planarity within a given embedding in terms of a forest embedded in the face-vertex incidence graph.
- We reduce the upward planarity testing problem to the one of finding a suitable orientation of a tree that synthetically represents the decomposition of a graph into its triconnected components.

- We show that the above combinatorial results yield an optimal $O(n)$-time upward planarity testing algorithm for single-source digraphs. The algorithm also constructs an upward planar drawing if the test is successful. Our algorithm improves over the previous best result [19] by an $O(n)$ factor in the time complexity.

- We efficiently parallelize the above algorithm to achieve $O(\log n)$ time on a CRCW PRAM with $n \log\log n / \log n$ processors. Hence, we provide the first efficient parallel algorithm for upward planarity testing. Our parallel time complexity is the same as the one of the best parallel algorithm for planarity testing [28, 27].

- Finally, as a side effect, we provide an optimal parallel algorithm for testing acyclicity of a planar n-vertex single-source-digraph in $O(\log n)$ time with $n/\log n$ processors on an EREW PRAM.

Open problems include:

- devising efficient dynamic algorithms for upward planarity testing of single-source digraphs;

- reducing the time complexity of upward planarity testing of planar digraphs with a prescribed embedding; and

- identifying additional classes of planar digraphs for which upward planarity can be tested in polynomial time.

The rest of this paper is organized as follows. Section 2 contains preliminary definitions and results. The problem of testing upward planarity for planar single-source digraphs with a prescribed embedding is investigated in Section 3. A combinatorial characterization of upward planarity for single-source digraphs is given in Sections 4, 5, and 6. The complete upward planarity testing algorithm for single-source digraphs is presented in Section 7.

2 Preliminaries

In this section we recall some terminology and basic results on upward planarity. We also review the SPQR-tree, introduced in [10, 11], and the combinatorial characterization of upward planarity for embedded planar digraphs, shown in [3, 4]. We assume familiarity with planar graphs.

2.1 Drawings and Embeddings

A drawing of a graph maps each vertex to a distinct point of the plane and each edge (u, v) to a simple Jordan curve with endpoints u and v. A *polyline* drawing maps each edge into a polygonal chain. A *straight-line* drawing maps each edge into a straight-line segment.

A drawing is planar if no two edges intersect, except, possibly, at common endpoints. A graph is planar if it has a planar drawing. Two planar drawings of a planar graph G are equivalent if, for each vertex v, they have the same circular clockwise sequence of edges incident on v. Hence, the planar drawings of G are partitioned into equivalence classes. Each such class is called an *embedding* of G. An embedded planar graph is a planar graph with a prescribed embedding. A triconnected planar graph has a unique embedding, up to a reflection. A planar drawing divides the plane into topologically connected regions delimited by circuits, called faces. The external face is the boundary of the unbounded region. Two drawings with the same embedding have the same faces. Hence, one can speak about the faces of an embedding.

Let G be a digraph (i.e., a directed graph). A *source* of G is a vertex without incoming edges. A *sink* of G is a vertex without outgoing edges. An *internal vertex* of G has both incoming and outgoing edges. Let f be a face of planar drawing (or embedding) of a digraph. A *source-switch* (*sink-switch*) of f is a source (sink) of f. Note that a *source-switch* (*sink-switch*) is not necessarily a source (sink) of G.

An *upward* drawing of a digraph is such that all the edges are represented by directed curves increasing monotonically in the vertical direction. A digraph has an upward drawing if and only if it is acyclic. A digraph is *upward planar* if it admits a planar upward drawing. Note that a planar acyclic digraph does not necessarily have a planar upward drawing, as shown in Fig. 1.b. An upward planar digraph also admits a planar upward straight-line drawing [21, 9]. A planar st-digraph is a planar digraph with exactly one source s and one sink t, connected by edge (s,t). A digraph is upward planar if and only if it is a subgraph of a planar st-digraph [21, 9]. If a digraph has a single source, then it is upward planar if and only if its biconnected components are upward planar [19].

A planar embedding of a digraph is *candidate* if the incoming (outgoing) edges around each vertex are consecutive. The planar embedding underlying an upward drawing is candidate. An *upward embedding* of a digraph G is an embedding of G where each angle formed by pairs of incoming or outgoing edges is assigned a label in the set $\{$*small*, *large*$\}$, such that there exists a planar straight-line upward drawing of $\vec{G}$ where each angle labeled *small* has measure $< \pi$ and each angle labeled *large* has measure $> \pi$.

2.2 SPQR-Trees

In the following we summarize SPQR-trees. For more details, see [10, 11]. SPQR-trees are closely related to the classical decomposition of biconnected graphs into triconnected components [17].

Let G be a biconnected graph. A *split pair* of G is either a separation-pair or a pair of adjacent vertices. A *split component* of a split pair $\{u,v\}$ is either an edge (u,v) or a maximal subgraph C of G such that $\{u,v\}$ is not a split pair of C.

Let $\{s,t\}$ be a split pair of G. A *maximal split pair* $\{u,v\}$ of G with respect to $\{s,t\}$ is such that for any other split pair $\{u',v'\}$, vertices u, v, and t are in the same split component.

Let $e(s,t)$ be an edge of G, called *reference edge*. The *SPQR-tree* $\mathcal{T}$ of G with respect to e describes a recursive decomposition of G induced by its split pairs. Tree $\mathcal{T}$ is a rooted ordered tree whose nodes are of four types: S, P, Q, and R. Each node μ of $\mathcal{T}$ has an associated biconnected multigraph, called the *skeleton* of μ, and denoted by *skeleton*(μ). Also, it is associated with an edge of the skeleton of the parent ν of μ, called the *virtual edge* of μ in *skeleton*(ν). Tree $\mathcal{T}$ is recursively defined as follows.

Trivial Case: If G consists of exactly two parallel edges between s and t, then $\mathcal{T}$ consists of a single Q-node whose skeleton is G itself.

Parallel Case: If the split pair $\{s,t\}$ has at least three split components $G_1, \cdots, G_k$ ($k \geq 3$), the root of $\mathcal{T}$ is a P-node μ. Graph *skeleton*(μ) consists of k parallel edges between s and t, denoted $e_1, \cdots, e_k$, with $e_1 = e$.

Series Case: Otherwise, the split pair $\{s,t\}$ has exactly two split components, one of them is the reference edge e, and we denote with G' the other split component. If G' has cutvertices $c_1, \cdots, c_{k-1}$ ($k \geq 2$) that partition G into its blocks $G_1, \cdots, G_k$, in this order from s to t, the root of $\mathcal{T}$ is an S-node μ. Graph *skeleton*(μ) is the cycle $e_0, e_1, \cdots, e_k$, where $e_0 = e$, $c_0 = s$, $c_k = t$, and e_i connects c_{i-1} with c_i ($i = 1 \cdots k$).

Rigid Case: If none of the above cases applies, let $\{s_1, t_1\}, \cdots, \{s_k, t_k\}$ be the maximal split pairs of G with respect to $\{s, t\}$ ($k \geq 1$), and for $i = 1, \cdots, k$, let G_i be the union of all the split components of $\{s_i, t_i\}$ but the one containing the reference edge e. The root of $\mathcal{T}$ is an R-node μ. Graph *skeleton*(μ) is obtained from G by replacing each subgraph G_i with the edge e_i between s_i and t_i.

Except for the trivial case, μ has children $\mu_1, \cdots, \mu_k$ in this order, such that μ_i is the root of the SPQR-tree of graph $G_i \cup e_i$ with respect to reference edge e_i ($i = 1, \cdots, k$). Edge e_i is said to be the *virtual edge* of node μ_i in *skeleton*(μ) and of node μ in *skeleton*(μ_i). Graph G_i is called the *pertinent graph* of node μ_i, and of edge e_i.

The tree $\mathcal{T}$ so obtained has a Q-node associated with each edge of G, except the reference edge e. We complete the SPQR-tree by adding another Q-node, representing the reference edge e, and making it the parent of μ so that it becomes the root. Observe that we are defining SPQR-trees of graphs, however the same definition can be applied to digraphs. An example of SPQR-tree is shown in Fig. 2.

Figure 2: (a) A planar biconnected digraph G. (b) SPQR-tree $\mathcal{T}$ of G and skeletons of the R-nodes.

Let μ be a node of $\mathcal{T}$. We have:

- if μ is an R-node, then *skeleton*(μ) is a triconnected graph;
- if μ is an S-node, then *skeleton*(μ) is a cycle;
- if μ is a P-node, then *skeleton*(μ) is a triconnected multigraph consisting of a bundle of multiple edges;
- if μ is a Q-node, then *skeleton*(μ) is a biconnected multigraph consisting of two multiple edges;

The skeletons of the nodes of $\mathcal{T}$ are homeomorphic to subgraphs of G. The SPQR-trees of G with respect to different reference edges are isomorphic and are obtained one from the other by selecting a different Q-node as the root. Hence, we can define the *unrooted SPQR-tree* of G without ambiguity.

The SPQR-tree $\mathcal{T}$ of a graph G with n vertices and m edges has m Q-nodes and $O(n)$ S-, P-, and R-nodes. Also, the total number of vertices of the skeletons stored at the nodes of $\mathcal{T}$ is $O(n)$.

A graph G is planar if and only if the skeletons of all the nodes of the SPQR-tree $\mathcal{T}$ of G are planar. An SPQR-tree $\mathcal{T}$ rooted at a given Q-node represents all the planar drawing of G having the reference edge (associated to the Q-node at the root) on the external face (see Fig. 2). Namely, such drawings can be constructed by the following recursive procedure:

- construct a drawing of the skeleton of the root ρ with the reference edge of the parent of ρ on the external face;
- for each child μ of ρ
  - let e be the virtual edge of μ in *skeleton*(ρ), and let H be the pertinent graph of μ plus edge e;
  - recursively draw H with the reference edge e on the external face;
  - in *skeleton*(ρ), replace virtual edge e with the above drawing of H minus edge e.

2.3 Upward Planarity Testing of Embedded Digraphs

In the rest of this section we recall the combinatorial characterization of upward planarity for planar digraphs with a fixed embedding, given in [3, 4], which will be extensively used in this paper.

In [3, 4], the problem of testing whether an embedded planar digraph G admits a planar upward drawing is formulated as a perfect c-matching problem on a bipartite graph derived from G. To introduce this formulation, we need some notation and definitions.

Let Γ be a planar straight-line upward drawing of an embedded upward planar digraph G. (As shown in [21, 9], every upward planar digraph admits a planar upward straight-line drawing.) We say that a sink t (source s) of G is *assigned* to a face f of Γ if the angle defined by the two edges of f incident on t (s) is greater than π. Informally speaking, t (s) is assigned to f if it "penetrates" into face f. Clearly, each sink (source) can be assigned only to one face, while an internal vertex is not assigned to any face. In [3, 4], it is shown that the number of vertices assigned to a face f in any upward drawing is equal to the capacity $c(f)$ of the face itself, which is defined as follows. Let $2n_f$ be the number of sink-switches and source-switches of f. We set $c(f) = n_f - 1$ if f is an internal face, and $c(f) = n_f + 1$ if f is the external face. In the following, we associate to each vertex x and each face f the quantity $Z(x, f)$, where $Z(x, f) = 1$ if x is a switch of f, $Z(x, f) = 0$ otherwise.

This intuitive idea of assignment of vertices to faces can be formally expressed as a perfect c-matching problem [24]. Namely, given a planar digraph G with a candidate planar embedding Ψ, we associate with G and Ψ the bipartite network $N(L_1, L_2, A)$ with vertex set $L_1 \cup L_2$ and edge-set A, where: (i) the vertices of L_1 represent the sources and sinks of G; (ii) the vertices of L_2 represent the faces of Ψ; and (iii) A has an edge (v, f) if and only the vertex of G represented by $v \in L_1$ lies on the face of Ψ represented by $f \in L_2$. The c-matching problem for G and Ψ is described by the following equations.

$$\sum_{(v,f)\in A} x_{vf} = c(f), \forall f \in L_2$$

$$\sum_{(v,f)\in A} x_{vf} = 1, \forall v \in L_1$$

where $x_{vf} = 1$ indicates that vertex v is *assigned* to face f and $x_{vf} = 0$ indicates otherwise. A solution of this c-matching problem is called an *upward consistent assignment* of the variables x_{vf} and is denoted by $\mathcal{A}$. The equations of the first set are called *capacity equations*.

Lemma 1 *[3, 4] Let G be a digraph with a candidate planar embedding Ψ. Then Ψ is an upward embedding of G if and only if the c-matching problem associated with G and Ψ admits an upward consistent assignment.*

If $\mathcal{A}$ is an upward consistent assignment for Ψ, and f is a face of Ψ, we denote with $A(f)$ the set of vertex of G assigned to f in $\mathcal{A}$.

3 Embedded Digraphs

In this section we give a new combinatorial characterization of upward planarity for planar single-source-digraphs with a prescribed embedding. This characterization yields an optimal algorithm for testing whether an embedded planar single-source-digraph has an upward planar drawing that preserves the embedding.

Given a planar single-source-digraph G and an upward embedding Γ of G, from the first condition on the capacity equations of the perfect matching problem and from the fact that G has a unique source, the following properties can be easily derived (see Fig. 3):

Figure 3: Schematic illustration of: (a) the external face, and (b) an internal face of an upward planar drawing of a planar single-source-digraph.

Fact 1 *The source s of G is the bottommost vertex of Γ.*

Fact 2 *For the external face h of Γ, all the sink-switches are sinks of G and are assigned to h. (See Fig. 3.a.)*

Fact 3 *For each internal face f, at most one sink-switch (the topmost vertex of f in Γ) is not a sink of G and all but one sink switches are assigned to f. (See Fig. 3.b.)*

We shall also use the following result about cycles in planar single-source-digraphs:

Lemma 2 *Let G and G' be planar single-source-digraphs such that G' is obtained from G by means of one of the following operations:*

- *adding a new vertex v and a new edge (u, v) or (v, u), connecting v to a vertex u of G;*
- *adding a directed edge between the source and a sink on the same face in some embedding of G;*
- *adding a directed edge between two sink-switches on the same face in some embedding of G;*

Then G is acyclic if and only if G' is acyclic.

Proof: The acyclicity is trivially preserved by the first two operations. Regarding the third operation, consider an embedding of G with the source s on the external face, and assume, for a contradiction, that G is acyclic and adding the edge (t', t'') between sink-switches t' and t'' of face f causes the resulting graph G' to have a cycle γ (see Fig. 4). Cycle γ must consist of edge (t', t'') and a directed path π' in G from t'' to t'. Let v be the neighbor of t'' in f inside γ, and let π'' be a directed path from the source s of G to v. Since s is external to cycle γ, path π'' must have at least a vertex in common with path π'. Let u be the last vertex of π'' that is also on π'. We have the G has a cycle consisting of edge (v, t''), the subpath of π' from t'' to u, and the subpath of π'' from u to v, a contradiction. □

Figure 4: Illustration of the proof of Lemma 2.

Given an embedded planar single-source-digraph G, the *face-sink graph* F of G is the incidence graph of the faces and the sink-switches of G (see Figs. 5.a and 6). Namely:

- The vertices of F are the faces and the sink-switches of G.
- Graph F has an edge (f, v) if v is a sink-switch on face f.

Theorem 1 *Let G be an embedded planar single-source-digraph and h a face of G. Digraph G has an upward planar drawing that preserves the embedding and with external face h if and only if all the following conditions are satisfied:*

Figure 5: (a) An embedded planar single-source-digraph G and its face-sink graph. (b–c) Upward drawings of G that preserve the embedding, with different external faces.

Figure 6: An embedded planar single-source-digraph that does not have an upward drawing that preserves the embedding.

1. *graph F is a forest;*
2. *a tree T of F has no internal vertices of G, while the remaining trees have exactly one internal vertex;*
3. *h is in tree T; and*
4. *the source of G is in the boundary of h.*

Also, if G has such a drawing, then an embedded planar st-digraph G' containing G as an embedded subgraph is obtained as follows:

1. *root tree T at h and each remaining tree of F at its (unique) internal vertex;*
2. *orient F by directing edges towards the roots;*
3. *prune the leaves from every tree of F; and*
4. *add the resulting forest $\hat{F}$ and the edge (s, h) to G.*

Proof: *Only If.* Let Γ be any planar upward drawing of G that preserves the original planar embedding and has external face h. By Fact 1, Condition 4 is verified. Orient the face-sink graph F of G by directing edge (v, f) from v to f if v is a sink assigned to face f, and from f to v otherwise. By Facts 2–3, each vertex of F has at most one outgoing edge. Specifically, each internal face and each sink has exactly one outgoing edge, while each internal vertex and the external face have no outgoing edges. Now, label the vertices of F as follows: the label of a sink-switch is its y-coordinate in Γ; the label of an internal face f is $y(v) - \epsilon$, where v is the sink-switch not assigned to f, and ϵ is a suitably small positive real value; and the label of the external face h is $+\infty$. Since Γ is an upward drawing, the edges of F are directed by increasing labels. We conclude that F is a forest of sink-trees. One tree is rooted at h, while the other trees are rooted at internal vertices. This shows Conditions 1–2. Condition 3 follows from Fact 1. Additional geometric considerations show that adding $\hat{F}$ to G yields a planar st-digraph.

If. We show that, if F satisfies the conditions of the theorem, then G is a subgraph of a planar st-digraph G', which is obtained as the union of G and $\hat{F}$. This implies that G is upward planar. Planarity is preserved since a star is inserted in each face. Also, G' has exactly one source (s) and one sink (h) connected by a directed edge. It remains to be shown that G' is acyclic. By the construction of G' and Lemma 2, we have that G' is acyclic if and only if G is acyclic. Assume, for a contradiction, that G is not acyclic. Let γ be a cycle of G that does not enclose any other cycle. Note that the source s must be outside γ. If γ is a face of G, then F has an isolated vertex associated with face γ, a contradiction. Otherwise (γ is not a face of G), the subgraph $\hat{F}'$ of $\hat{F}$ enclosed by γ consists of a forest of trees, each with exactly one internal vertex. Let H be the digraph obtained from the subgraph of G enclosed by γ by removing the edges of γ, and adding a new vertex s' together with edges from s' to all the vertices of γ. By our choice of cycle γ, H is a planar single-source-digraph. Adding $\hat{F}'$ to H yields a planar single-source-digraph without sinks, and hence a digraph with cycles. By Lemma 2, H must also have cycles, again a contradiction. □

Theorem 1 is illustrated in Figs. 5–6. The following algorithm tests whether an embedded planar single-source-digraph G is upward planar, and reports all the faces of G that can be external in an upward planar drawing of G with the prescribed embedding.

Algorithm *Embedded-Test*

1. Construct the face-sink graph F of G.
2. Check Conditions 1 and 2 of Theorem 1. If these conditions are not verified, then return "not-upward-planar" and stop.
3. Report the set of faces of G that contain vertex s in their boundary and are associated with nodes of tree T. If such set of faces is empty, then return "not-upward-planar" else return "upward-planar".

For the example of Fig. 5, Algorithm *Embedded-Test* returns "upward-planar" and reports two faces.

Theorem 2 *Let G be an embedded planar single-source-digraph with n vertices. Algorithm* Embedded-Test *determines whether G has an upward planar drawing that preserves the embedding, and reports all the admissible external faces. It runs in $O(n)$ sequential time and in $O(\log n)$ time on a CRCW PRAM with $n \cdot \alpha(n)/\log n$ processors, using $O(n)$ space.*

Proof: The correctness of the algorithm follows directly from Theorem 1. All the steps can be performed sequentially in $O(n)$ time with straightforward methods.

Regarding the parallel complexity, Steps 1 and 3 take $O(\log n)$ time on a CREW PRAM with $n/\log n$ processors, using list-ranking [5]. Step 2 can be executed by computing a spanning forest of the face-sink graph, which takes $O(\log n)$ time on a CRCW PRAM with $n \cdot \alpha(n)/\log n$ processors [5], and thus determines the parallel time complexity. □

4 Upward Planarity and SPQR-Trees

Let G be a biconnected single-source-digraph. In this section we give a combinatorial characterization of the upward planarity of G using SPQR-trees.

4.1 Basic Definitions and Main Result

A digraph is *expanded* if each internal vertex has exactly one incoming edge or one outgoing edge. The *expansion* of a digraph is obtained by replacing each internal vertex v with two new vertices v_1 and v_2, which inherit the incoming and outgoing edges of v, respectively, and the edge (v_1, v_2). Observe that a digraph is acyclic if and only if its expansion is acyclic. A planar embedding of an expanded digraph is candidate. In the rest of this section we consider only expanded digraphs because of the following property.

Fact 4 *A digraph is upward planar if and only if its expansion is upward planar.*

Let $G' = (V', E')$ and $G'' = (V'', E'')$ be two digraphs. The *union* of G' and G'', denoted by $G' \cup G''$, is a digraph $G = (V, E)$ with $V = V' \cup V''$ and $E = E' \cup E''$, i.e. G is obtained from G' and G'' by identifying the vertices in V' and V'' with common labels.

Let G be a biconnected digraph and let $\{u, v\}$ be a separation pair of G that decomposes G into p split components $J_1, \ldots, J_p$. We call *component separated by* $\{u, v\}$ or simply *component* any digraph obtained as the union of q of the split components in $\{J_1, \ldots, J_p\}$, with $0 < q < p$.

We call *peak* a digraph consisting of three vertices a, b, and t, and two directed edges (a, t) and (b, t). See Fig. 4.1.b.

Figure 7: (a) Directed edge, (b) peak, (c) valley, and (d) zig-zag.

Let K be a component of G with respect to the separation pair $\{u, v\}$. In the following, we denote with $G - K$ the digraph obtained from G by deleting every vertex belonging to K, except for the vertices u and v, and by K^0 the digraph obtained from K by deleting vertices u and v. Also, for simplicity, we write $G - K_1 - K_2 \cdots - K_m$ instead of $(\cdots((G - K_1) - K_2) \cdots - K_m)$

Finally, we associate to a component K of G either a peak or a directed edge (see Fig. 4.1.a), according to the following rules:

Rule 1. u and v are sources of K: a peak with $a \equiv u$ and $b \equiv v$.

Rule 2. u is a source of K and v is a sink of K:

- **a.** $s \notin K^0$: a directed edge (u, v).
- **b.** $s \in K^0$: a peak with $a \equiv u$ and $b \equiv v$.

Rule 3. u is a source of K and v is an internal vertex of K:

- **a.** v is a source of $G - K$ and $s \notin K^0$: a directed edge (u, v).
- **b.** v is not a source of $G - K$ or $s \in K^0$: a peak with $a \equiv u$ and $b \equiv v$.

Rule 4. u and v are not sources of K:

- **a.** u is a source of $G - K$: a directed edge (u, v).
- **b.** u is not a source of $G - K$: a directed edge (v, u).

The digraph associated to K by the above rules is called *directed-virtual-edge*, and will be denoted by $d(K, G - K)$. Observe that the choice of the directed-virtual-edge depends in general both on K and on $G - K$.

We call *minor* of G either G itself or the digraph

$$G - K_1 - \ldots - K_m \cup d(K_1, G - K_1) \cup \ldots \cup d(K_m, G - K_m),$$

where $K_1, \ldots, K_m$, $m \geq 1$, are components of G with the property that no two components share a common edge. In other words, the digraph $G - K_1 - \ldots - K_m \cup d(K_1, G - K_1) \cup \ldots \cup d(K_m, G - K_m)$ is obtained from G by replacing $K_1, \ldots, K_m$ with the corresponding directed-virtual-edge (see Fig. 8). Observe that, in general, a minor of a minor of G is not a minor of G.

Let G be a planar single-source-digraph, and let $\mathcal{T}$ be its SPQR-unrooted tree. The *$s\mathcal{T}$-skeleton* of a node μ of $\mathcal{T}$, denoted by $s\mathcal{T}$-skeleton(μ), is the minor of G obtained from the skeleton of μ by

Figure 8: Construction of a minor.

Figure 9: The directed skeletons of the R-nodes of the digraph of Fig. 2.

replacing each virtual edge with the directed-virtual-edge associated to its pertinent digraph. The *reference directed-virtual-edge* is the directed-virtual-edge associated with the pertinent digraph of the reference edge of μ. Examples of sT-skeletons are shown in Fig. 9.

The main result of this section is summarized in the following theorem:

Theorem 3 *A biconnected acyclic single-source-digraph G is upward planar if and only if there exists a rooting of the SPQR-tree $\mathcal{T}$ of the expansion of G at a reference edge containing the source such that the sT-skeleton of each node μ of $\mathcal{T}$ has a planar upward drawing with the reference directed virtual edge on the external face.*

The proof of Theorem 3 is given in the next two sections: Section 5 shows the *Only-If* part, while Section 6 shows the *If* part. We give here some preliminary lemmas that will be used in the next sections.

4.2 Preliminary lemmas

In the following S_G, T_G and I_G will denote the set of sources, sinks and internal vertices of G, respectively. If G has exactly one source, we denote such source by $s(G)$.

We will make use of the following operations defined on an edge $e = (x, y)$ of a digraph G:

- *Contraction* (denoted by G/e) transforms G into a digraph G' obtained from G by removing edge e and by identifying vertices x and y.
- *Direct Subdivision* transforms G into a digraph G' obtained from G by removing edge e and by adding a vertex z and edges (x, z) and (z, y).

We say that digraphs G_1 and G_2 are *homeomorphic* if both can be obtained by performing a finite number of direct subdivisions of a digraph G. Observe that we can have $G_1 = G$ or $G_2 = G$.

We call *valley* a digraph consisting of three vertices s', a and b, and two directed edges (s', a) and (s', b) (see Fig. 4.1.c). Also, we call *zig-zag* a digraph consisting of four vertices s', t, a and b, and three directed edges (s', t), (s', b) and (a, t) (see Fig. 4.1.d).

We denote by $x \to y$ a path from vertex x to vertex y. A vertex u is said to be *dominated* by vertex v if there is a path $v \to u$. We say that vertices x and y are *incomparable* in G, denoted by $x \parallel y$, if neither there exists in G a path $x \to y$, nor a path $x \to y$.

Lemma 3 *[19] If G is an sT-digraph with $u \parallel v$ in G, then there exists in G a subgraph homeomorphic to a valley, with $a \equiv u$ and $b \equiv v$.*

Lemma 4 *[19] Let G be a connected acyclic digraph with exactly two sources u and v. Then there exists in G a subgraph homeomorphic to a peak, with $a \equiv u$ and $b \equiv v$.*

Fact 5 *Let G be a biconnected digraph and let $\{u, v\}$ be a separation pair of G. Neither u nor v is a cut-vertex of any component of G with respect to $\{u, v\}$.*

Lemma 5 *Let G be a connected sT-digraph and let $\{u, v\}$ be a separation pair of G. Let K be a component with respect to $\{u, v\}$ such that $v \in I_K$ and K has exactly one source u. Then digraph K contains a subgraph homeomorphic to a peak, with $a \equiv u$ and $b \equiv v$.*

Proof: Let w be a vertex of K such that there exist two vertex disjoint directed paths $u \to w$ and $v \to w$ contained in K. The subgraph of K consisting of all edges and vertices belonging to the two paths is homeomorphic to a peak. Suppose w does not exist. Let V_v be the set of all vertices dominated by v in K (except v).

Let $\bar{V}_v$ be the set of all vertices not dominated by v in K. Observe that both V_v and $\bar{V}_v$ are nonempty and disjoint. Also, $V_v \cup \bar{V}_v \cup \{v\}$ is the set of vertices of K. Since v is not a cut-vertex of K, there is an edge connecting a vertex x of V_v with a vertex y of $\bar{V}_v$. If such an edge is (x, y), then y belongs to V_v, a contradiction. If such an edge is (y, x), then $x = w$, again a contradiction. $\square$

Finally the following fact and lemmas concerning the embeddings of G will be used in the proof of the main theorem.

Fact 6 *Let G be a digraph and let G' be a digraph homeomorphic to a subgraph G'' of G. We have that:*

(i) If G is acyclic then G' is acyclic.

(ii) If G is expanded then G' is expanded.

(iii) If G is upward planar, then G' is upward planar.

(iv) If G'' is an sT-digraph then G' is an sT-digraph.

Lemma 6 *[19] Let G be a digraph, let (u, v) be an edge of G, and let vertex u (v) have out-degree (in-degree) 1 in G. Let $G'' = G/(u, v)$; we have that:*

(i) If G is acyclic then G'' is acyclic.

(ii) If G is upward planar then G'' is upward planar.

Lemma 7 *Let G be an upward planar sT-digraph, and let Ψ_G be an upward embedding of G. Let $e = (s, u)$ be an edge of G embedded on the external face α of Ψ_G. Let $G' = G - e \cup P$ where P is a valley, i.e. P is the path $\{(s', s), (s', u)\}$. Then G' is an sT-digraph and has an upward embedding $\Psi_{G'}$, with P embedded on the external face.*

Proof: From Ψ_G we simply derive a candidate planar embedding $\Psi_{G'}$ of G', by replacing the edge (s, u) of G with the path P (see Fig. 10).

Figure 10: Construction of $\Psi_{G'}$.

We show now that $\Psi_{G'}$ is an upward embedding. This is done by deriving an upward consistent assignment $\mathcal{A}'$ associated to $\Psi_{G'}$ from the upward consistent assignment $\mathcal{A}$ associated to Ψ_G.

Let β be the internal face of Ψ_G containing the edge (s, u). We denote by $\gamma_1, \ldots, \gamma_p$ the faces of Ψ_G different from α and β.

Clearly, there is a one-to-one correspondence between the faces of Ψ_G and the faces of $\Psi_{G'}$. We denote by $\alpha', \beta', \gamma'_1, \ldots, \gamma'_p$ the faces of $\Psi_{G'}$ corresponding to $\alpha, \beta, \gamma_1, \ldots, \gamma_p$.

Also, it is $\gamma'_i = \gamma_i$, for $i = 1, \ldots, p$, and thus $c(\gamma'_i) = c(\gamma_i)$, for $i = 1, \ldots, p$. We show that $c(\alpha') = c(\alpha)$ and $c(\beta') = c(\beta)$.

We have that $c(\alpha') = c(\alpha) - Z(s, \alpha) + Z(s, \alpha') - Z(u, \alpha) + Z(u, \alpha') + Z(s', \alpha')$. Since the edge (s, u) of α has been replaced by the edge (s', u) of α', it is $Z(u, \alpha') = Z(u, \alpha)$. It is easy to see that $Z(s, \alpha) = 1$, $Z(s, \alpha') = 0$ and $Z(s', \alpha') = 1$. Thus, $c(\alpha') = c(\alpha)$. In the same fashion, we can prove that $c(\beta') = c(\beta)$.

Let S (S') and T (T') be the set of sources and sinks of G (G'), respectively. Clearly $S' = S - \{s\} \cup \{s'\}$, and $T' = T$. $\mathcal{A}'$ is derived from $\mathcal{A}$ in the following way:

- $A'(\gamma'_i) = A(\gamma_i)$, for $i = 1, \ldots, p$
- $A'(\alpha') = A(\alpha) - \{s\} \cup \{s'\}$, and
- $A'(\beta') = A(\beta)$.

It is straightforward to prove that $|A'(f)| = c(f)$ for each face $f \in \Psi_{G'}$. Since $\Psi_{G'}$ is a candidate embedding and $\mathcal{A}'$ is upward consistent, by Lemma 1 G' is upward planar. $\square$

5 Proof of Necessity for Theorem 3

Lemma 8 *Let G be a planar expanded sT-digraph G, $\{u, v\}$ a separation pair of G, and K a component with respect to $\{u, v\}$. Let $H = G - K$ and let $d_K = d(K, H)$ be the directed-virtual-edge associated to K with respect to G. Finally, let H' be the minor $H \cup d_K$. We have:*

(i) H' is an expanded, acyclic sT-digraph.

(ii) If G is upward planar then H' is upward planar.

(iii) If G has an upward embedding and $s(G) \in K^0$, then H' has an upward embedding with d_K on the external face.

Proof: The following cases are possible, each corresponding to one of Rules 1-4.

1. u and v are sources of K.

   By Lemma 4 there is in K a path P_K homeomorphic to a peak, with $a \equiv u$ and $b \equiv v$ (see Fig. 11). Thus, $H' = H \cup d_K$ is homeomorphic to a subgraph of G.

Figure 11: Various cases in the proof of Lemma 8.

   (i) By Fact 6 H' is an expanded, acyclic digraph. We show now that it contains only one source. Vertices u and v are not both sources in H, otherwise G would contain two sources. Suppose one of u, v (say u) is a source of H. Then $u \equiv s(G)$ and $u \equiv s(H')$, thus u is the only source of H'. Suppose neither u nor v is a source in H, then $s(G) \in H^0$ and s is the only source of H'.

   (ii) By Fact 6 H' is upward planar.

2a. u is a source of K and v is a sink of K and $s \notin K^0$.

   Since $s \notin K^0$, u is the only source of K and there exists in K a path $u \to v$. Thus there is in K a path P_K homeomorphic to the directed edge (u, v) and then there is in G a subgraph homeomorphic to $H' = H \cup d_K$ (see Fig. 11).

(i) By Fact 6 H' is an acyclic, expanded digraph. Because of the existence of edge (u, v), v is not a source of H'. If $u \in S_H$, then $s(G) \equiv u$, thus u is the only source of H'. If $u \notin S_H$, $s(G) \in H^0$, and $s(G) \equiv s(H')$.

(ii) By Fact 6 H' is upward planar.

2b. u is a source of K and v is a sink of K and $s \in K^0$.

Since $s \in K^0$, $s \neq u$ and by Lemma 4 there is a vertex t distint from s and u, and two vertex disjoint paths $s \rightarrow t$ and $u \rightarrow t$. Note that u is not a source of H, otherwise u is a source of G, a contradiction. So v is the only source of H and since there is in G a path $s \rightarrow v$, we have that there is a path $s \rightarrow v$ in K. Moreover, since v is the only source of H, there is a path $v \rightarrow u$ in H and thus there is a path $v \rightarrow u$ in G. This implies that the path $s \rightarrow v$ and the path $u \rightarrow t$ are disjoint, otherwise there would be a path $u \rightarrow v$, and G would contain a cycle, a contradiction.

Let P_1 and P_2 be a path $s \rightarrow v$ and a path $s \rightarrow t$, respectively. Let z be the last vertex common to paths P_1 and P_2. Note that $z \neq t$, since P_1 and any path $u \rightarrow t$ are disjoint. Thus there exists in K a path P_K homeomorphic to a zig-zag, and then there is in G a subgraph homeomorphic to $H'' \equiv H \cup (z, v) \cup (z, t) \cup (u, t)$ (see Fig. 11).

Observe that in H'', vertex v has in-degree 1. If we contract the edge (z, v) we obtain $H''/(z, v) \equiv H \cup d_K = H'$.

(i) By Fact 6 H'' is acyclic and expanded, thus by Fact 6 H' is acyclic. Note that every vertex except for v has the same in- and out- degrees in H'' and H'. Since H'' is expanded and v is a source of H', H' is an expanded digraph. Moreover, since v is the only source of H, v is the only source of H'.

(ii) By Fact 6 H'' is upward planar, thus by Fact 6 H' is upward planar.

3a. u is a source of K and v is an internal vertex of K, v is a source of $G - K$ and $s \notin K^0$.

Since u is the only source of K, there is a path $u \rightarrow v$ in K and the proof is analogous to the one of Case 2.a.

3b. u is a source of K and v is an internal vertex of K, v is not a source of $G - K$ or $s \in K^0$.

1 If $s \in K^0$, then $s \neq u$ and K contains two sources: the proof is as in Case 2.b.

2 If u is the only source of K, by Lemma 5, K contains a path P_K homeomorphic to a peak, with $u \equiv a$ and $v \equiv b$, and $H' = H \cup d_K$ is homeomorphic to a subgraph of G.
(i) By Fact 6 H' is an expanded, acyclic digraph. Moreover, since v is not a source of H, vertex u is the only source of H', and H' is an sT-digraph.
(ii) By Fact 6 H' is upward planar.

4. u and v are not sources of K.

Since u and v are not sources of K, we have that $s(G) \in K^0$. Then either u or v (or both) is a source of H. We discuss only the case when u is a source of H and Rule 4.a is applied. In fact, if u is not a source of H, then v is a source of H and Rule 4.b is applied, that is, the roles of u and v are interchanged. Three cases are possible.

1) If there exists a path $u \rightarrow v$ in K, then there is in K a path P_K homeomorphic to the directed edge (u, v) (see Fig. 11).
(i) By Fact 6 H' is an expanded acyclic digraph. Because of the existence of the edge (u, v), u is the only source of H'.
(ii) By Fact 6 H' is upward planar.

2) If u and v are incomparable in K, by Lemma 3 there exists in K a path P_K homeomorphic to a valley with $a \equiv u$ and $b \equiv v$ (see Fig. 11). Thus $H'' \equiv H \cup (s,v) \cup (s,u)$ is homeomorphic to a subgraph of G. Observe that in H'', vertex u has in-degree 1. If we contract the edge (s,u) we obtain $H''/(s,u) \equiv H \cup d_K = H'$.

   (i) By Fact 6 H'' is acyclic and expanded, thus by Lemma 6 H' is acyclic. Note that all vertices but u have the same in- and out- degrees in H'' and H'. Since H'' is expanded and u is a source of H', then H' is an expanded digraph. Moreover, since u is the only source of H, then u is the only source of H'.

   (ii) By Fact 6 H'' is upward planar, thus by Lemma 6 H' is upward planar.

3) If there exists a path P_K from v to u in K, then v is also a source of H (see Fig. 11). Otherwise u would be the only source of H, so there would be a path $u \rightarrow v$ in H and thus a path $u \rightarrow v$ in G, a contradiction. So both u and v are sources of H.

   (i) Since H is an expanded, acyclic digraph, and since both vertex u and vertex v are sources of H, $H' = H \cup (u,v)$ is expanded and acyclic. Furthermore, it contains only one source u.

   (ii)Let H'' be the digraph $H \cup (v,u)$. Digraph H'' is homeomorphic to a subgraph of G and thus, by Fact 6, there exists an upward embedding $\Psi_{H''}$ of H'', with (v,u) on the external face. Furthermore, v is the only source of H'', and H'' is an expanded sT-digraph.

   H' can be obtained from H'' by the means of the following operations:

   1 Construct $\bar{H}$ from H'' by adding a vertex s' and replacing the edge (v,u) with the valley $\{(s',v),(s',u)\}$.

   2 Construct H' from $\bar{H}$ by contracting the edge (s',u).

   By Lemma 7, $\bar{H}$ has an upward embedding with the valley $\{(s',v),(s',u)\}$ on the external face. By Lemma 6, H' has an upward embedding with the edge (u,v) on the external face.

□

Following the developments of the previous proof, for each case a particular path P_K is detected in the component K. These paths and the corresponding cases will play a central role in the next section.

The previous lemma refers to a particular minor of G obtained by replacing exactly one component, but it can be easily extended to any minor of G. Before proving next lemma, we have to observe a property of sources in minors. Suppose u is a source of G. Then u is a source in every component of G that contains u. By a simple inspection of Rules 1-4 we have the following:

Fact 7 *Let G be a digraph and K be a component of G with respect to the separation pair $\{u,v\}$. Let G' be a minor of G such that $K \subset G'$. If u is a source of $G-K$, then u is a source in $G'-K$.*

Lemma 9 *Let G be a directed expanded sT-digraph and let $\tilde{H}' = G - K_1 - \ldots - K_m \cup d_{K_1} \ldots \cup d_{K_m}$ be a minor of G, where $d_{K_i} = d(K_i, G-K_i)$, then*

(i) $\tilde{H}'$ is an expanded, acyclic sT-digraph

(ii) If G is upward planar, then $\tilde{H}'$ is upward planar.

(iii) If G is upward planar and $s(G) \in K_i^0$, $1 \leq i \leq m$, then H' has an upward embedding with d_{K_i} on the external face.

Proof: The proof is by induction. When $m = 1$, the minor is $\tilde{H}' = G - K_1 \cup d(K_1, G - K_1)$, and by Lemma 8 the basis of the induction holds. Now, suppose that the minor $\tilde{G} = G - K_1 - \ldots - K_{l-1} \cup d_{K_1} \cup \ldots \cup d_{K_{l-1}}$ is an expanded, acyclic sT-digraph and is upward planar. We show that the minor $\tilde{J}' = G - K_1 - \ldots - K_l \cup d_{K_1} \cup \ldots \cup d_{K_l}$ is an expanded sT-digraph and is upward planar.

Note first that $\tilde{J}' = \tilde{G} - K_l \cup d_{K_l}$. If we can prove that $d_{K_l} = d(K_l, G - K_l)$, is equal to $d(K_l, \tilde{G} - K_l)$, then the thesis follows by Lemma 8. In other words we have to prove that the directed-virtual-edge which substitutes K_l remains the same when Rules 1-4 are applied to the pair $(K_l, \tilde{G} - K_l)$ rather than to the pair $(K_l, G - K_l)$.

When Rules 1 and 2 are applied, the directed-virtual-edge depends only on the component K_l, and $d_K = d(K_l, G - K_l) = d(K_l, \tilde{G} - K_l)$. Hence we have to consider only Rules 3 and 4.

3.a v is a source of $G - K_l$ and $s(G) \notin K_l^0$. By fact 7, v is a source in $\tilde{G} - K_l$. Moreover, since $s(G) \notin K_l^0$ we have that $s(\tilde{G}) \notin K_l^0$ and Rule 3.a must be applied to the pair $(K_l, \tilde{G} - K_l)$.

3.b If $s \in K_l^\circ$, Rule 3.b must be applied to the pair $(K_l, \tilde{G} - K_l)$.

Suppose that v is a not a source of $G - K_l$ and $s \notin K_l^\circ$. Note that v is a sink in $G - K_l$ (v cannot be internal in $G - K_l$ since v is internal in K_l and G is expanded).

If v is not a source of $\tilde{G} - K_l$ then Rule 3.b must be applied to the pair $(K_l, \tilde{G} - K_l)$.

Suppose v is a source of $\tilde{G} - K_l$, then there exists a component J of $G - K_l$, with $J \notin \tilde{G} - K_l$, having v and u_J as poles, whose directed-virtual-edge is either a peak or the edge (v, u_J). Observe that v is a sink of J. If $s \notin J^\circ$ then u_J is the only source of J. Hence Rule 2.a is applied to the pair $(J, G - J)$ and $d(J, G - J)$ is the edge (u_J, v), a contradiction.

If $s \in J^\circ$ then u_J is a source of $G - J$ (since v is internal in $K_l \in G - J$). Thus u_J is not a source of J, otherwise G contains two sources. Then Rule 4.a is applied to the pair $(J, G - J)$ and $d(J, G - J)$ is the edge (u_J, v), a contradiction.

4.a u is a source of $G - K_l$, then u is a source of $\tilde{G} - K_l$.

4.b Since u is not a source in $G - K_l$ and $s(G) \in K_l$, then v is a source in $G - K_l$ and v is a source in $\tilde{G} - K_l$. So Rule 4.a can be applied to the pair $(K_l, \tilde{G} - K_l)$, with v and u interchanged.

□

The proof of the necessity of Theorem 3 is now a simple corollary of Lemma 9. In fact, for each node μ of tree $\mathcal{T}$, the sT-skeleton of μ is a minor of G.

6 Proof of Sufficiency for Theorem 3.

Lemma 10 *Let G be a planar expanded sT-digraph, $\{u, v\}$ a separation pair of G, and K a component with respect to $\{u, v\}$ such that $s(G) \in K$. Let $H = G - K$ and let $d_K = d(K, H)$ and $d_H = d(H, K)$ be the directed-virtual-edges associated to K and H (with respect to G), respectively. Finally, let H' be the minor $H \cup d_K$ and K' be the minor $K \cup d_H$. If K' is upward planar and H' has an upward embedding with d_K on the external face, then G is upward planar.*

Before proving the above lemma, we need some preliminary results, namely the following Lemmas 11–15.

We remind the reader that in the proof of Lemma 8, in correspondence with each case of the proof, a path P_K is detected in the component K. Such path P_K will be used in the following lemma (see Fig. 12).

Lemma 11 *Let G be a planar expanded sT-digraph G, $\{u,v\}$ a separation pair of G, and K a component with respect to $\{u,v\}$. Let $H = G - K$ and let $d_K = d(K,H)$ be the directed-virtual-edge associated to K with respect to G. Let H' be the minor $H \cup d_K$ and let $\bar{H} = H \cup P_K$. Suppose H' has an upward embedding $\Psi_{H'}$, with d_K embedded on the external face if $s(G) \in K^0$. Denote by $\Psi_H \subset \Psi_{H'}$ the upward embedding of H contained in $\Psi_{H'}$ and let α_H be the face of Ψ_H in which d_K is embedded. We have that:*

(i) $\bar{H} = H \cup P_K$ *is an expanded, acyclic digraph.*

(ii) $\bar{H}$ *is an sT-digraph and has an upward embedding $\Psi_{\bar{H}}$, with $\Psi_H \subset \Psi_{\bar{H}}$ and P_K embedded in face γ of Ψ_H.*

Proof: Since $\bar{H}$ is a subgraph of G then it is expanded and acyclic and (i) holds.

We prove now (ii). Since both d_K and P_K depend on the component K, we distinguish the following four cases, each corresponding to the cases of the proof of Lemma 8.

Figure 12: Illustration of the statement of Lemma 11.

1. P_K is homeomorphic to d_K, hence $\bar{H}$ is homeomorphic to H'. By Fact 6 the lemma holds.

2. a) P_K is homeomorphic to d_K, hence $\bar{H}$ is homeomorphic to H'. By Fact 6 the lemma holds.

   b) Since $s \in K^0$, d_K is embedded on the external face of $\Psi_{H'}$. Observe that $\bar{H}$ can be obtained from H' substituting the edge (v,t) with a path homeomorphic to a valley. Thus, by Lemma 7 and by Fact 6 the lemma holds.

3. a) P_K is homeomorphic to d_K, hence $\bar{H}$ is homeomorphic to H'. By Fact 6 the lemma holds.

   b) If $s \in K^\circ$ then the proof is as in Case 2.b. If $s \notin K^\circ$ then P_K is homeomorphic to d_K, hence $\bar{H}$ is homeomorphic to H'. By Fact 6 the lemma holds.

4. 1) P_K is homeomorphic to d_K, hence $\bar{H}$ is homeomorphic to H'. By Fact 6 the lemma holds.

   2) Since $s \in K^0$, d_K is embedded on the external face of $\Psi_{H'}$. Observe that $\bar{H}$ can be obtained from H' substituting the edge (u,v) with a path homeomorphic to a valley. Thus, by Lemma 7 and by Fact 6 the lemma holds.

   3) Observe that $\bar{H}$ can be obtained from H' by reversing edge (u,v) and by direct subdivision. Thus, following the proof of Case 4.3 of Lemma 8, and by Fact 6, the lemma holds.

$\square$

Suppose K, H, K' and H' satisfy the conditions of Lemma 10. Let P_K and P_H be the paths associated with d_K and d_H (see Fig. 13.a, 13.b). Except for the common endpoints u and v, P_K and P_H are disjoint paths of G, since they lie in different components. Thus, $C = P_K \cup P_H$ is a simple (undirected) cycle of G.

Let $\bar{K} = K \cup P_H$ and $\bar{H} = H \cup P_K$ (see Fig. 13.c, 13.d). Since K' and H' are upward planar, by the previous lemma also $\bar{K}$ and $\bar{H}$ are upward planar. Let $\Psi_{\bar{K}}$ and $\Psi_{\bar{H}}$ be two upward embeddings of $\bar{K}$ and $\bar{H}$, respectively, and let $\alpha_{\bar{K}}$ and $\alpha_{\bar{H}}$ be the corresponding external faces. Now, let K^* (H^*) be the subgraph of $\bar{K}$ ($\bar{H}$) embedded inside C in $\Psi_{\bar{K}}$ ($\Psi_{\bar{H}}$) (see Fig. 13.e, 13.f). We have the following

Lemma 12 *Let s^* be a source of K^* (H^*). Then $s^* \in C$.*

Proof: Suppose $s^* \notin C$. Then s^* is a source of $\bar{K}$ embedded inside C in $\Psi_{\bar{K}}$. Since $\bar{K}$ is an sT-digraph, s^* is the only source of $\bar{K}$ and is embedded on the external face of every upward embedding of $\bar{K}$, and thus cannot be embedded inside C in $\Psi_{\bar{K}}$, a contradiction.

□

Figure 13: Construction of Ψ_G

Lemma 13 *The digraph K^* (H^*) is an expanded, acyclic sT-digraph.*

Proof: Since K^* is a subgraph of G then it is acyclic and expanded. By Lemma 12, all of the sources of K^* belong to $C = P_K \cup P_H$. In order to prove the sT-property, we have to examine P_K and P_H. If both P_K and P_H are homeomorphic to the edge (u, v) ((v, u)), then $s(K^*)$ is either u or v. Suppose now that P_K is not homeomorphic to an edge. We consider the following cases.

1. P_K homeomorphic to a peak. Clearly, if P_H is homeomorphic to (u, v) or (v, u) or the valley, then C has only one source. We show now that P_H is not homeomorphic to a peak or to a zig-zag. Since P_K is a peak, then K is in case 1 or in case 3b.2 of the proof of Lemma 8.

   Suppose P_H is a peak. Then H is in case 1 or in case 3b.2.

   a) P_K as in case 1. If H is in case 1, then G contains two sources, a contradiction. If H is in case 3b.2, then v is not a source of K, a contradiction.

   b) P_K as in case 3b.2. If H is in case 1, then the proof is as above. If H is in case 3b.2, then v is internal both in H and in K, a contradiction.

   Suppose P_H is homeomorphic to a zig-zag. Then H is in case 2.b or in case 3b.1.

   a) P_K as in case 1. If H is in case 2.b or in case 3.b1, then $s(G) \in H^\circ$. Since u is source both in K and in H then G contains two sources, a contradiction.

   b) P_K as in case 3b.2. The proof is as above.

2. P_K homeomorphic to a zig-zag. Clearly, if P_H is homeomorphic to (v, u), then C has only one source. If P_H is homeomorphic to a peak, the proof is as for the case P_H homeomorphic to a zig-zag and P_K homeomorphic to a peak. We show now that P_H is not homeomorphic to the edge u, v, to a valley, or to the zig-zag. Since P_K is a zig-zag, then K is in case 2.b or in case 3b.1.: thus $s(G) \in K^\circ$ and u is a source of K.

   Suppose P_H is homeomorphic to the edge (u, v). Then H is in case 2.a or in case 3.a or in case 4.1. If H is in case 2.a or in case 3.a, then u is a source of G and G contains two sources, a contradiction. If H is in case 4.1, then $s(G) \in H^\circ$, a contradiction.

   Suppose P_H is homeomorphic to a valley. Then H is in case 4.2 and $s(G) \in H^\circ$, a contradiction.

   Suppose P_H is homeomorphic to a zig-zag. Then H is in case 2.b and in case 3b.1 and $s(G) \in H^\circ$, a contradiction.

3. P_K homeomorphic to a valley. Clearly, if P_H is homeomorphic to the edge (u, v) or to the edge (v, u) or to the a peak, then C has only one source. If P_H is homeomorphic to a zig-zag, the proof is as for the case P_H homeomorphic to a valley and P_K homeomorphic to a zig-zag. We show now that P_H is not homeomorphic to a valley.

   Since both P_K and P_H are homeomorphic to a valley, then both K and H are in case 4.2: thus $s(G) \in K^\circ$ and $s(G) \in H^\circ$, a contradiction.

□

The proof of next lemma can be found in [3, 4].

Lemma 14 *Let G be digraph, let Ψ_G be a candidate planar embedding of G and let face $\delta \in \Psi_G$. Finally, let $\mathcal{A}$ be an assignment of the sinks and the sources of G to the faces of Ψ_G. If $|A(f)| = c(f)$ for each face $f \in \Psi_G$, with $f \neq \delta$, then $|A(\delta)| = c(\delta)$.*

Suppose G, K, H, K' and H' satisfy the conditions of Lemma 10. In the following we give a constructive procedure to derive an embedding Ψ_G of G from two upward embeddings $\Psi_{H'}$ and $\Psi_{K'}$ of H' and K', respectively. We then show Ψ_G to be upward.

Let $\bar{K}$, $\bar{H}$, K^*, H^*, Ψ_{K^*}, Ψ_{H^*} be as defined for Lemma 13 and let $\Psi_{K^*} \subseteq \Psi_{\bar{K}}$ ($\Psi_{H^*} \subseteq \Psi_{\bar{H}}$) be the upward embedding of K^* (H^*) contained in $\Psi_{\bar{K}}$ ($\Psi_{\bar{H}}$). Denote by α_{K^*} and α_{H^*} the external faces of Ψ_{K^*} and Ψ_{H^*}, respectively (see Fig. 13.e–f). Recall that $\alpha_{K^*} = \alpha_{H^*} = C$.

Let $G^* = K^* \cup H^*$ (see Fig. 13.g). Ψ_{G^*} is a planar embedding of G^* such that $\Psi_{K^*} \subset \Psi_{G^*}$, $\Psi_{H^*} \subset \Psi_{G^*}$, and P_K and P_H lie on the external face α_{G^*} of Ψ_{G^*} (i.e. $\alpha_{G^*} = C$).

We denote by γ_{G^*} the other face of Ψ_{G^*} (besides α_{G^*}) containing both edges of K^* and edges of H^*.

Lemma 15 *Let G^* and Ψ_{G^*} be defined as above. Then:*

(i) G^* is an expanded, acyclic sT-digraph.

(ii) Ψ_{G^*} is an upward embedding.

Proof: (i) Since G^* is a subgraph of G, then G^* is expanded and acyclic. Since G^* is acyclic, it contains at least one source. Suppose G^* contains two sources, s_1 and s_2. Since K^* (H^*) is an sT-digraph, s_1 and s_2 cannot belong both to K^* (H^*). Thus, w.l.o.g., $s_1 \in K^*$ and $s_2 \in H^*$. Furthermore, by Lemma 12 s_1 and s_2 lie on cycle C, so they are contained both in K^* and H^*, a contradiction. In the following, we denote by s^* the source of G^*.

(ii) We derive an upward consistent assignment $\mathcal{A}_{G^*}$ from the upward consistent assignments $\mathcal{A}_{K^*}$ and $\mathcal{A}_{H^*}$ associated with Ψ_{K^*} and Ψ_{H^*}.

Note first that $A_{K^*}(\alpha_{K^*}) = A_{H^*}(\alpha_{H^*})$. In fact, let $T*$ be the set of sink-switches of $C = \alpha_{K^*} = \alpha_{H^*}$. Since Ψ_{K^*} and Ψ_{H^*} are upward embeddings of sT-digraphs, by Fact 2, each sink-switch on α_{K^*} and α_{H^*} is a sink of K^* and H^*, respectively, and they are all assigned to α_{H^*} and α_{K^*}. We have that $A_{K^*}(\alpha_{K^*}) = A_{H^*}(\alpha_{H^*}) = T^* \cup \{s^*\}$.

For each face $f \in \Psi_{G^*}$, with $f \neq \gamma_{G^*}$ and $f \neq \alpha_{G^*}$, we have that f belongs to Ψ_{K^*} or f belongs to Ψ_{H^*} but not to both. It is trivial to see that the following assignment to the faces of $\Psi_{G^*} - \{\gamma_{G^*}\}$ is feasible (i.e., the number of vertices assigned to each face equals the capacity of the face):

- $A_{G^*}(\alpha_{G^*}) = A_{K^*}(\alpha_{K^*}) = A_{H^*}(\alpha_{H^*})$
- $A_{G^*}(f) = A_{K^*}(f)$, for $f \neq \alpha_{G^*}$ and $f \in \Psi_{K^*}$
- $A_{G^*}(f) = A_{H^*}(f)$, for $f \neq \alpha_{G^*}$ and $f \in \Psi_{H^*}$.

Observe that all the sinks of Ψ_{G^*} not assigned by the above assignment lie on face γ_{G^*} and so they can be assigned to it in $\mathcal{A}_{G^*}$. Since Ψ_{G^*} is a candidate embedding, and for each face $f \in \Psi_{G^*} - \{\gamma_{G^*}\}$ it is $|A_{G^*}(f)| = c(f)$, by Lemma 14 we have that $A_{G^*}(\gamma_{G^*}) = c(\gamma_{G^*})$. Thus, A_{G^*} is an upward consistent assignment and G^* is upward planar.

□

We are now able to give the proof of Lemma 10.

Proof: (of Lemma 10) A planar embedding Ψ_G of G can be obtained from the upward embedding $\Psi_{\bar{K}}$ and $\Psi_{\bar{H}}$ in such a way that $\Psi_{G^*} \subseteq \Psi_G$. Each face of Ψ_G belongs either to $\Psi_{\bar{K}}$ or to $\Psi_{\bar{H}}$, except for two faces which share edges both of $\bar{K}$ and $\bar{H}$. One of these two faces coincides with face γ_{G^*} of Ψ_{G^*}, and thus it is internal in Ψ_{G^*}: we denote it by γ_G; we denote the other face by β_G (see Fig. 13.h).

We derive now an assignment $\mathcal{A}_G$ from the upward consistent assignments $\mathcal{A}_{\bar{K}}$, $\mathcal{A}_{\bar{H}}$ and $\mathcal{A}_{G^*}$, in the following way:

- $A_G(\gamma_G) = A_{G^*}(\gamma_{G^*})$
- $A_G(f) = A_{\bar{K}}(f)$, for $f \in \Psi_{\bar{K}}$
- $A_G(f) = A_{\bar{H}}(f)$, for $f \in \Psi_{\bar{K}}$
- It is easy to see that all remaining sinks and (eventually) the source of G stay on face β_G and so they are assigned to β_G in $\mathcal{A}_G$.

In order to prove that A_G is upward consistent we have to show that:

(i) every sink and the source of G is assigned to exactly one face of Ψ_G,

(ii) every internal vertex of G is not assigned to any face of Ψ_G,

(iii) the number of vertices assigned to each face equals the capacity of the face.

Observe first that since P_K is embedded in the external face $\alpha_{\bar{H}}$ of $\Psi_{\bar{H}}$, $\alpha_{\bar{H}}$ is not a face of Ψ_G.

We first prove (i). It is easy to see that each sink (source) of G is assigned to at least one face. We have to prove that each sink (source) of G is assigned to at most one face. Let x be a vertex assigned to two faces f_1 and f_2 in $\mathcal{A}_G$ ($f_1 \neq f_2$). From the definition of $\mathcal{A}_G$, $f_1 \in \Psi_{\bar{K}}$ and $f_2 \in \Psi_{\bar{H}}$. This is not possible if $x \neq u$ or $x \neq v$. If $x = u$ ($x = v$) then it is assigned to the external face $\alpha_{\bar{H}} = f_2$ of $\Psi_{\bar{H}}$ in $\mathcal{A}_{\bar{H}}$ and thus $f_2 \notin \Psi_G$, a contradiction.

(ii) Let x be an internal vertex of G and suppose it is assigned to a face f of Ψ_G. Again $x = u$ or $x = v$. Assume w.l.o.g. $x = u$. Observe that f is not a face of $\Psi_{\bar{H}}$, so f is a face of $\Psi_{\bar{H}}$. Furthermore, $f \neq \gamma_G$, since neither u nor v is assigned to it in $\mathcal{A}_G$.

Suppose $x = v$ is a sink in $\bar{K}$ and v is not a sink in G.

Suppose now v is assigned to $f \neq \beta_{\bar{K}}$. This implies that both $\beta_{\bar{K}}$ and $\gamma_{\bar{K}}$ are internal faces of $\Psi_{\bar{K}}$ and hence that $s(\bar{K}) = s(G) \in \bar{K}$.

Since v is a sink of $\bar{K}$, then P_H has an incoming edge into v, and P_H is homeomorphic to the edge (u, v) or to a zig-zag or to a valley. If P_H is homeomorphic to edge (u, v) then H is in case 2.a or 3.a or 4.1 of the proof of Lemma 8. In case 2.a, v is a sink of H and thus u a sink of G, a contradiction. In case 3.a v is a source of K and thus it is not a sink in $\bar{K}$, a contradiction. If H is in case 4.1 then $s \in H^\circ$, a contradiction.

If P_H is homeomorphic to a valley, then H is in case 4.2 and $s(G) \in H^\circ$, a contradiction.

If P_H is homeomorphic to a zig-zag, then H is in case 2.b or 3.b2. In both cases, $s(G) \in H^\circ$, a contradiction.

(iii) Since G is an expanded digraph, every planar embedding is candidate. By Lemma 14 the capacity equation for face β_G is satisfied, and $\mathcal{A}_G$ is upward consistent. $\square$

Lemma 10 refers to digraph G. We extend the result to a minor $\tilde{G}$ of G. This is done by showing that, under certain restrictions, the directed-virtual-edges substituting components can be chosen independently one from another. In particular, this is true if the source s of G belongs to its minor. Note that in this case a minor of a minor is a minor.

Lemma 16 *Let $\tilde{G}$ be a minor of G such that $s(\tilde{G}) = s(G)$. Let $\{u, v\}$ be a split pair of $\tilde{G}$ such that $\{u, v\}$ is also a split pair of G. Let $\tilde{K}$ be a component of $\tilde{G}$ w.r.t. $\{u, v\}$ and let K be the corresponding component of G, i.e. K is obtained from $\tilde{K}$ by replacing each directed virtual edge of $\tilde{K}$ with its associated component of G. Then it is $d(\tilde{K}, \tilde{G} - \tilde{K}) = d(K, G - K)$.*

Proof: In the following we denote by H the digraph $G - K$ and by $\tilde{H}$ the digraph $\tilde{G} - \tilde{K}$. Observe that, for each component J of G such that $J \notin \tilde{G}$ (i.e. J is substituted by its directed-virtual-edge), since $s(G) \in \tilde{G}$, then $s(G) \notin J^\circ$.

We examine the following four cases corresponding to the substitution Rules 1-4 applied to components K and $\tilde{K}$.

1. Since u and v are sources of K, by Fact 7 u and v are sources also of $\tilde{K}$. Then $d(\tilde{K}, \tilde{H})$ is a peak and the lemma holds.

2.a. u is source of K, v is sink of K and $s(G) \notin K^\circ$. By Fact 7 u is source of $\tilde{K}$. Since $s(\tilde{G}) = s(G)$ then $s(G) \notin \tilde{K}^\circ$. We show now that v is a sink of $\tilde{K}$. In fact v is a sink of all the components having v as a pole. By the substitution rules whenever a component $J \in K$ has v as a sink then the associated directed-virtual-edge has an edge incoming into v except for Rules 2.b. and 4.b. But in both cases $s(G) \in J^\circ$, contradicting that $s \notin K_\circ$. Then Rule 2.a is applied to $\tilde{K}$ and $d(\tilde{K}, \tilde{H})$ is the edge (u, v).

2.b. u is source of K, v is sink of K and $s \in K^\circ$. By Fact 7 u is source of $\tilde{K}$. Furthermore $s(\tilde{G}) \in \tilde{K}^\circ$.

If v is a sink of $\tilde{K}$ then Rule 2.b is applied to $\tilde{K}$ and $d(\tilde{K}, \tilde{H})$ is a peak.

If v is internal of $\tilde{K}$ then Rule 3.b is applied to $\tilde{K}$ and $d(\tilde{K}, \tilde{H})$ is a peak.

If v is a source of $\tilde{K}$ then $\tilde{K}$ has three sources and the minor $\tilde{G}$ is not an sT-digraph.

3.a. u is source of K, v is internal of K, $s \in H$ and v is source of H. By Fact 7 u is a source of $\tilde{K}$ and v is source of $\tilde{H}$.

If v is internal in $\tilde{K}$ then Rule 3a is applied to $\tilde{K}$ and $d(\tilde{K}, \tilde{H})$ is the edge (u, v).

If v is a sink of $\tilde{K}$ then Rule 2a is applied to $\tilde{K}$ and $d(\tilde{K}, \tilde{H})$ is the edge (u, v).

If v is a source of $\tilde{K}$ then v is source of $\tilde{G}$. Since $s(\tilde{G}) = s(G) \neq v$ then $\tilde{G}$ has two sources, a contradiction.

3.b. u is source of K, v is internal in K.

By Fact 7 u is a source of $\tilde{K}$. We remind the reader that two cases are possible:

(i) $s(G) \in K^\circ$. Then v is the only source of H. In fact if u is a source of H then u is a source of G, a contradiction. By Fact 7 v is a source of $\tilde{H}$.
If v is internal in $\tilde{K}$ then Rule 3b is applied to $\tilde{K}$ and $d(\tilde{K}, \tilde{H})$ is a peak.
If v is a sink of $\tilde{K}$ then Rule 2b is applied to $\tilde{K}$ and $d(\tilde{K}, \tilde{H})$ is a peak.
If v is a source of $\tilde{K}$ then v is a source of $\tilde{G}$. Since $s(\tilde{G}) = s(G) \neq v$ then $\tilde{G}$ has two sources, a contradiction.

(ii) v is not a source of H and $s(G) \in H$. Since G is expanded, v is a sink of H.

If v is a source of $\tilde{K}$ then Rule 1 is applied to $\tilde{K}$ and $d(\tilde{K}, \tilde{H})$ is a peak.

If v is internal in $\tilde{K}$ and v is not a source of $\tilde{H}$ then Rule 3.b is applied to $\tilde{K}$ and $d(\tilde{K}, \tilde{H})$ is a peak. If v is a source of $\tilde{H}$, there exists a component J of H, with $J \notin \tilde{H}$, having v and u_J as poles, whose directed-virtual-edge is either a peak or the edge (v, u_J). Observe that v is a sink of J and u_J is the source of J (since $s(G) \notin J^\circ$). But in this case Rule 2.a is applied to J and $d(J, G-J)$ is the edge (u_J, v), a contradiction.

If v is a sink of $\tilde{K}$, since v is internal in K, there exists a component J, with $J \in K$ and $J \notin \tilde{K}$ (i.e. J has been replaced by its directed-virtual-edge), having $v_J = v$ and u_J as poles, whose directed-virtual-edge is the edge (u_J, v). Furthermore, v is internal or a source of J. If v is a source of J then $d(J, G-J)$ is not the edge (u_J, v). If v is internal in J then Rules 3.a or 4.b must be applied in order to have $d(J, G-J) = (u_J, v)$. Rule 3.a implies that v is a source of $G - J$, hence v is a source of K, a contradiction. Rule 4.a implies that $s(G) \in J^\circ$, a contradiction.

4. u and v are not sources of K. $s \in K^\circ$. Either u or v (or both) is a source of H. Suppose, w.l.o.g., that u is a source of H. u is not a source of $\tilde{K}$, otherwise the minor $\tilde{G}$ has two sources.

   If v is not a source of $\tilde{K}$, by Fact 7 u is a source of $\tilde{H}$, Rule 4.a is applied and $d(\tilde{K}, \tilde{H})$ is the edge (u, v).

   If v is a source of $\tilde{K}$, there exists a component J of K, with $J \notin \tilde{K}$, having $v_J = v$ and u_J as poles, such that v is a non-source of J, and its directed-virtual-edge is either a peak or the edge (v, u_J). Observe that, since $s(G) \notin J^\circ$, u_J is the source of J. If v is a sink of J, Rule 2.a is applied to J and $d(J, G-J)$ is the edge (u_J, v), a contradiction. If v is internal in J, Rule 3.b is applied and, since $s(G) \notin J^\circ$, v is a sink of $G - J$ and hence a sink of $K - J$. Hence v is not a source of the minor $K - J \cup d(J, G-J)$. Let Z be a component of K, with $Z \neq J$. Since v is a sink of $K - J$ then v is a sink of Z. If $Z \in \tilde{K}$ then v is not a source of $\tilde{K}$, a contradiction. If $Z \notin \tilde{K}$ then $d(Z, G-Z) \in \tilde{K}$. Since u_Z is a source of Z ($s(G) \notin Z^\circ$) then Rule 2.a is applied, $d(Z, G-Z)$ is the edge (u_Z, v) and v is not a source of $\tilde{K}$, a contradiction.

□

We are now able to prove the sufficiency part of Theorem 3.

Proof: (of sufficiency of Theorem 3) Let $\mathcal{T}$ be the rooted $SPQR$-tree associated with the graph G, and let $\mu_1, \ldots, \mu_m$ be the sequence of nodes of $\mathcal{T}$ deriving from a DFS visit of T, starting at its root. Let $Skel(\mu_i)$, $i = 1, \ldots, m$ be the sT-skeleton associated with $\mu_1, \ldots, \mu_m$.

For each node μ_i, let d_i^c be the directed virtual edge of μ_i in the skeleton associated with the parent of μ_i, and let d_i^p be the directed virtual edge of the parent of μ_i in $Skel(\mu_i)$. Clearly, if $\mu_i = \mu_1$ is the root, then $d_i^p = d_1^p = \{\emptyset\}$. Finally let $\tilde{G}_i = Skel(\mu_1) - d_2^c \cup (Skel(\mu_2) - d_2^p) - d_3^c \cup (Skel(\mu_3) - d_3^p) - \ldots - d_i^c \cup (Skel(\mu_i) - d_i^p)$.

We show that $\tilde{G}_i$ is a minor of G, with $s(G) \in \tilde{G}_i$, and that $\tilde{G}_i$ is upward planar.

For $i = 1$ we have that $\tilde{G}_1 = Skel(\mu_1)$ and the claim trivially holds. Suppose that $\tilde{G}_{l-1}$ is a minor of G with $s(G) \in \tilde{G}_{l-1}$, and that $\tilde{G}_{l-1}$ is upward planar, we show that $\tilde{G}_l$ is a minor of G, with $s(G) \in \tilde{G}_l$ and that $\tilde{G}_l$ is upward planar.

Let H be the pertinent graph of d_l^c and K be the pertinent graph of d_l^p. Recall that $K \cup H = G$ and that K and H share exactly two vertices. Moreover $d_l^p = d(K, H)$ and $d_l^c = d(H, K)$.

Let $J_1, \ldots, J_q$ be components of G contained in K, such that

$$\tilde{G}_{l-1} = G - H - J_1 - \ldots - J_q \cup d(H, K) \cup d(J_1, G - J_1) \cup \ldots \cup d(J_q, G - J_q) =$$

$$K - J_1 - \ldots - J_q \cup d(H,K) \cup d(J_1, G - J_1) \cup \ldots \cup d(J_q, G - J_q)$$

and let $Z_1, \ldots, Z_r$ be split components of G contained in H, such that

$$Skel(\mu_l) = G - K - Z_1 - \ldots - Z_r \cup d(K,H) \cup d(Z_1, G - Z_1) \cup \ldots \cup d(Z_r, G - Z_r) =$$

$$H - Z_1 - \ldots - Z_q \cup d(K,H) \cup d(Z_1, G - Z_1) \cup \ldots \cup d(Z_r, G - Z_r)$$

.

The digraph $\tilde{G}_l = K \cup H - J_1 - \ldots - J_q - Z_1 - \ldots - Z_q \cup d(J_1, G - J_1) \cup \ldots \cup d(J_q, G - J_q) \cup d(Z_1, G - Z_1) \cup \ldots \cup d(Z_r, G - Z_r)$ is a minor of G. In fact, since $J_i \in K$, $i = 1, \ldots, q$ and $Z_t \in H$, $t = 1, \ldots, r$ it follows that J_i and Z_t do not share any edge, for $i = 1, \ldots, q$ and $t = 1, \ldots, r$.

Since $s(G) \in \tilde{G}_{l-1}$ then $s(G) \in \tilde{G}_l$.

Let $\tilde{K} = \tilde{G}_{l-1} - d_l^c$ and $\tilde{H} = Skel(\mu_l) - d_l^p$. Clearly $\tilde{G}_l = \tilde{K} \cup \tilde{H}$. By Lemma 16, $d(\tilde{K}, \tilde{H}) = d(K,H) = d_l^p$ and $d(\tilde{H}, \tilde{K}) = d(H,K) = d_l^c$. Since $\tilde{K} \cup d(\tilde{H}, \tilde{K}) = \tilde{G}_{l-1}$ is upward planar and $\tilde{H} \cup d(\tilde{K}, \tilde{H}) = Skel(\mu_l)$ is upward planar with $d(\tilde{K}, \tilde{H}) = d_l^p$ embedded on the external face then, by Lemma 10, $\tilde{G}_l$ is upward planar.

We show now that $\tilde{G}_m = G$. By induction $\tilde{G}_m$ is a minor of G. Suppose $\tilde{G}_m \neq G$, then there exists a component J of G such that $J \notin \tilde{G}_m$ and $d(J, G - J) \in \tilde{G}_m$. Let μ_j be the node of $\mathcal{T}$ such that $d(J, G - J) \in Skel(\mu_j)$. $d(J, G - J)$ is associated either with the parent of μ_j or with one of the children of μ_j. Since the tree $\mathcal{T}$ has been entirely visited then $d(J, G - J)$ has been substituted, and thus $d(J, G - J) \notin \tilde{G}_m$, a contradiction. □

7 Algorithm for General Single-Source Digraphs

Let G be a biconnected single-source-digraph. In this section we present an algorithm for testing whether G is upward planar.

Algorithm *Test*

1. Construct the expansion G' of G.
2. Test whether G' is planar. If G' is not planar, then return "not-upward-planar" and stop, else, construct an embedding for G'.
3. Test whether G' is acyclic. If G' is not acyclic, then return "not-upward-planar" and stop.
4. Construct the SPQR-tree $\mathcal{T}$ of G' and the skeletons of its nodes.
5. For each virtual edge e of a skeleton, classify each endpoint of e as a source, sink, or internal vertex in the pertinent digraph of e. Also, determine if the pertinent digraph of e contains the source.
6. For each node μ of $\mathcal{T}$, compute the sT-skeleton of μ.
7. For each R-node μ of $\mathcal{T}$:
   (a) Test whether the sT-skeleton of μ is upward planar by means of algorithm *Embedded-Test*. If *Embedded-Test* returns "not-upward-planar", then return "not-upward-planar" and stop.

(b) Mark the virtual edges of the skeleton of μ whose endpoints are on the external face in some upward drawing of the sT-skeleton of μ.

(c) For each unmarked virtual edge e of the skeleton of μ, constrain the tree edge associated with e to be directed towards μ.

(d) If the source is not in *skeleton*(μ), let ν be the node neighbor of μ whose pertinent digraph contains the source, and constrain the tree edge (μ, ν) to be directed towards ν.

8. Determine whether $\mathcal{T}$ can be rooted at a Q-node in such a way that orienting edges from children to parents satisfies the constraints of Steps 7c–7d. If such a rooting exists then return "upward-planar", else return "not-upward-planar".

For single-source-digraphs that are not biconnected we apply the above algorithm to each biconnected component.

Theorem 4 *Upward planarity testing of a single-source-digraph with n vertices can be done in $O(n)$ time using $O(n)$ space.*

Proof: Steps 1 and 3 can be trivially performed in $O(n)$ time. Planarity testing in Step 2 can also be done in $O(n)$ time [18]. The construction of the SPQR-tree and the skeletons of its nodes (Step 4) takes time $O(n)$ using a variation of the algorithm of [17]. The preprocessing of Step 5 consists essentially of a visit of $\mathcal{T}$, and can be done in $O(n)$ time. Let n_μ be the number of vertices of the skeleton of μ. The information collected in Step 5 allows to perform Step 6 in $O(n)$ time and Step 7d in $O(n_\mu)$ time. By Theorem 2, Step 7a takes $O(n_\mu)$ time. The output of Step 7a allows to perform Steps 7b and 7c in $O(n_\mu)$ time. Since $\sum_\mu n_\mu = O(n)$, the total complexity of Step 7 is $O(n)$. Finally, Step 8 consists of a visit of $\mathcal{T}$ and takes $O(n)$ time. $\square$

To parallelize Algorithm *Test*, we need an efficient way of testing in parallel whether a planar single-source-digraph with n vertices is acyclic. For this purpose, we can use the algorithm of [20], which runs in $O(\log^3 n)$ time on a CRCW PRAM with n processors. However, the particular structure of planar single-source-digraphs allows us to perform this test optimally. The following characterization is inspired by some ideas in [20].

Let G be an embedded expanded planar single-source-digraph. The *clockwise subgraph* of G is obtained by taking the first incoming edge of each internal vertex, in the clockwise order. The *counterclockwise subgraph* of G is similarly obtained by taking the first incoming edge of each internal vertex, in the counterclockwise order. Such subgraphs of G have all vertices with indegree 1 or 0.

Theorem 5 *An embedded expanded single-source-digraph G is acyclic if and only if both the clockwise and counterclockwise subgraphs of G are acyclic.*

Proof: The only-if part is trivial. For the if part, assume for a contradiction that G is not acyclic, and consider an arbitrary drawing of G with the prescribed embedding and with the source on the external face. We will show the existence of a cycle in either the clockwise or counterclockwise subgraph. Let γ be a cycle of G that does not enclose any other cycle. Since the source of G must be outside γ, all the edges incident on vertices of γ and inside γ must be outgoing edges. Hence, γ is contained in the clockwise or counterclockwise subgraph, depending on whether it is a clockwise or counterclockwise cycle. $\square$

The structure of each connected component of the clockwise and counterclockwise subgraphs is either a source-tree, or a collection of source trees with their roots connected in a directed cycle. Hence, one can test whether such subgraphs are acyclic using standard parallel techniques. Since expansion preserves acyclicity, we have:

Theorem 6 *Given an embedded planar single-source-digraph G with n vertices, one can test if G is acyclic in $O(\log n)$ time with $n/\log n$ processors on an EREW PRAM.*

By applying the result of Theorem 6 and various parallel techniques (in particular [14, 28, 27]) we can efficiently parallelize Algorithm *Test*:

Theorem 7 *Upward planarity testing of a single-source-digraph with n vertices can be done in in $O(\log n)$ time on a CRCW PRAM with $n \log\log n/\log n$ processors, using $O(n)$ space.*

As a consequence of Theorems 1 and 3, Algorithm *Test* can be easily extended such that if the n-vertex digraph G is found to be upward planar, a planar st-digraph G' with $O(n)$ vertices is constructed that contains G as a subdigraph. Hence, by applying the planar polyline upward drawing algorithm of Di Battista, Tamassia and Tollis [12] to G' and then removing the vertices and edges of G' that are not in G, we obtain a planar polyline upward drawing of G.

Theorem 8 *Algorithm* Test *can be extended so that it constructs a planar polyline upward drawing if the digraph is upward planar. The complexity bounds stay unchanged.*

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