Sum Amusements: A Case Study from the Analysis of Algorithms

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Abstract

While analyzing the average performance of an algorithm for a geometric problem, the summation $\sum_{k=1}^{n} k^m/2^k$ was encountered. This sum is a particular case of the sum $S_{m,n} = \sum_{k=1}^{n} \alpha^k k^m$ for $0 < \alpha < 1$. Solving $S_{m,n}$ requires a variety of techniques and is an interesting case study in the analysis of algorithms.

This paper examines both the solution of this summation and the methods used to obtain the solution. Three approaches to solving $S_{m,n}$ are examined. It is found that $S_{m,n}$ is defined by

$$S_{m,n} = C_m - D_{m,n},$$

where C_m is a function of α and m and $D_{m,n}$ is the product of α^n with a polynomial in n of degree m. Both functions are defined recursively.

Several techniques used in the analysis of algorithm are applied to evaluate these recurrences. Induction is used to prove results obtained by examining small cases. Generating function methods are employed to derive expressions defining C_m and $D_{m,n}$.

Asymptotics are found for C_m, $D_{m,n}$, and $S_{m,n}$. Integration is used to find a good asymptotic expression for C_m. Complex analysis techniques are employed to find an asymptotic for $D_{m,n}$. Asymptotics for $S_{m,n}$ are found by studying the sum in the region about the value of k that maximizes $k^m\alpha^k$.

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1. Introduction

In the analysis of algorithms, it is informative not only to look at the result of a summation, but to study the techniques used to find the result. While analyzing the average performance of an algorithm for a geometric problem, the sum

$$\sum_{k=1}^{n} \frac{k^m}{2^k}$$

was encountered. This sum is a particular case of the more general sum

$$S_{m,n} = \sum_{k=1}^{n} \alpha^k k^m.$$

$S_{m,n}$ has proved to be a good case study of techniques used in the mathematical analysis of algorithms.

Gauthier has considered the form of the solution for $S_{m,n}$ and derived methods for calculating the solution by recursion on m or n [1]. For the special case of $\alpha = 1$, $S_{m,n}$ reduces to the sum of the mth power of the first n integers. This case has received extensive coverage in the literature [2, 4].

In Section 2 we explore three different approaches to evaluate $S_{m,n}$. First, we generate solutions for small m in the hope of guessing a general expression that we can prove by induction. Second, we try two different transformation techniques, both of which allow us to express $S_{m,n}$ as a recurrence relation. For the third approach, we try a method called perturbation that produces the same recurrence relation. Observing once again the small cases and using the recurrence relation we are able to express $S_{m,n}$ in terms of the sequences C_m and $D_{m,n}$, both of which are defined recursively. C_m is a function of α and m. $D_{m,n}$ is the product of α^n with a polynomial in n of degree m whose coefficients are functions of α.

In Section 3 we analyze C_m. $D_{m,n}$ is analyzed in Section 4. For both sequences we examine small cases to find expressions for the sequences that we can prove using induction. We also apply generating function techniques to find expressions for the sequences. Finally, we find an asymptotic for C_m using integration and an asymptotic for $D_{m,n}$ using complex analysis techniques.

In Section 5 we look at the value of $S_{m,n}$ and its asymptotic behavior. We consider the relative growth rates of m and n. Asymptotic expressions are derived for three relative growth rates.

2. Solution of $S_{m,n}$

We will try several methods to evaluate $S_{m,n}$. We first attempt to derive directly a closed form for several values of m in the hope of guessing a general closed form that could be proved using induction. We then turn our attention to transformation methods. Both Abel's transformation and summation by parts give us a recurrence relation for $S_{m,n}$ that we can use to further evaluate the sum. Finally, we try a method called perturbation that also produces the same recurrence relation.

One approach for finding closed form solutions for summations of this type is by differentiation. We start with the geometric progression

$$S_{0,n} = \sum_{k=1}^{n} \alpha^k = \frac{\alpha - \alpha^{n+1}}{1-\alpha}.$$

Differentiating with respect to α, and then multiplying through by α gives us

$$\alpha \frac{d}{d\alpha} S_{0,n} = \alpha \frac{d}{d\alpha} \sum_{k=1}^{n} \alpha^k = \alpha \sum_{k=1}^{n} k\alpha^{k-1} = \sum_{k=1}^{n} k\alpha^k = S_{1,n}.$$

Thus we find that

$$\sum_{k=1}^{n} k\alpha^k = \alpha \frac{d}{d\alpha} \frac{\alpha - \alpha^{n+1}}{1-\alpha} = \frac{n\alpha^{n+2} - (n+1)\alpha^{n+1} + \alpha}{(1-\alpha)^2}.$$

This expression has a removable singularity at $\alpha = 1$ and the expression can be evaluated for all α. For any $m \geq 0$, we can in general differentiate $S_{m,n}$ and multiply by α to get $S_{m+1,n}$,

$$\alpha \frac{d}{d\alpha} S_{m,n} = \alpha \frac{d}{d\alpha} \sum_{k=1}^{n} k^m \alpha^k = \alpha \sum_{k=1}^{n} k^{m+1} \alpha^{k-1} = \sum_{k=1}^{n} k^{m+1} \alpha^k = S_{m+1,n}.$$

Unfortunately, this process is unwieldy even for small values of m. Repeated application reveals no apparent closed form solution which we could try to prove by induction.

Abel's transformation

$$\sum_{k=1}^{n} u_k v_k = u_{n+1} \sum_{k=1}^{n} v_k - \sum_{k=1}^{n} \left(\Delta u_k \sum_{p=1}^{n} v_p \right)$$

can be used to find more information about the sum. Letting $u_k = k^m$ and $v_k = \alpha^k$, we get

$$S_{m,n} = \sum_{k=1}^{n} k^m \alpha^k = (n+1)^m \sum_{k=1}^{n} \alpha^k - \sum_{k=1}^{n} \big((k+1)^m - k^m\big) \sum_{p=1}^{k} \alpha^p .$$

Evaluating the geometric progressions of α and then factoring out $\alpha/(1-\alpha)$ gives

$$\begin{aligned} S_{m,n} &= (n+1)^m \frac{\alpha - \alpha^{n+1}}{1-\alpha} - \sum_{k=1}^{n} \big((k+1)^m - k^m\big) \frac{\alpha - \alpha^{k+1}}{1-\alpha} \\ &= \frac{\alpha}{1-\alpha} \left((n+1)^m (1-\alpha^n) - \sum_{k=1}^{n} \big((k+1)^m - k^m\big)(1-\alpha^k) \right) . \end{aligned}$$

We can multiply the two factors in the summation and apply the associative law to break this into three parts,

$$S_{m,n} = \frac{\alpha}{1-\alpha} \left((n+1)^m (1-\alpha^n) - \sum_{k=1}^{n} (k+1)^m + \sum_{k=1}^{n} k^m + \sum_{k=1}^{n} \alpha^k \big((k+1)^m - k^m\big) \right) .$$

The net result of the first two summations is $-(n+1)^m + 1$. We can use the binomial theorem to evaluate $k^{m+1} - k^m$,

$$\begin{aligned} S_{m,n} &= \frac{\alpha}{1-\alpha} \left((n+1)^m (1-\alpha^n) - (n+1)^m + 1 + \sum_{k=1}^{n} \alpha^k \left(\sum_{j=0}^{m} \binom{m}{j} k^j - k^m \right) \right) \\ &= \frac{\alpha}{1-\alpha} \left(1 - \alpha^n (n+1)^m + \sum_{k=1}^{n} \alpha^k \sum_{j=0}^{m-1} \binom{m}{j} k^j \right) \\ &= \frac{\alpha}{1-\alpha} \left(1 - \alpha^n (n+1)^m + \sum_{k=1}^{n} \sum_{j=0}^{m-1} \binom{m}{j} k^j \alpha^k \right) . \end{aligned}$$

Changing the order of summation and recognizing $S_{j,n}$ gives a recurrence relation for $S_{m,n}$,

$$\begin{aligned} S_{m,n} &= \frac{\alpha}{1-\alpha} \left(1 - \alpha^n (n+1)^m + \sum_{j=0}^{m-1} \binom{m}{j} \sum_{k=1}^{n} k^j \alpha^k \right) \\ &= \frac{\alpha}{1-\alpha} \left(1 - \alpha^n (n+1)^m + \sum_{j=0}^{m-1} \binom{m}{j} S_{j,n} \right) . \end{aligned} \tag{1}$$

Another method we can use to analyze $S_{m,n}$ is summation by parts,

$$\sum_{k=1}^{n}(a_{k+1}-a_k)b_k \;=\; a_{n+1}b_{n+1}-a_1b_1-\sum_{k=1}^{n}(b_{k+1}-b_k)a_{k+1}.$$

To evaluate $S_{m,n}$ we have $a_{k+1}-a_k=\alpha^k$ and $b_k=k^m$, which implies that $a_1=0$ and $a_{k+1}=(\alpha-\alpha^{k+1})/(1-\alpha)$. Making these substitutions gives

$$S_{m,n} \;=\; \sum_{k=1}^{n}k^m\alpha^k \;=\; (n+1)^m\,\frac{\alpha-\alpha^{n+1}}{1-\alpha}-\sum_{k=1}^{n}\big((k+1)^m-k^m\big)\frac{\alpha-\alpha^{k+1}}{1-\alpha}.$$

This is the same intermediate result that was obtained using Abel's transformation.

As a final attempt to attack the sum $S_{m,n}$, we employ the method of perturbation [2]. First we perturb $S_{m,n}$ by changing the limits of summation,

$$S_{m,n} \;=\; \sum_{k=1}^{n}k^m\alpha^k \;=\; \sum_{k=0}^{n-1}(k+1)^m\alpha^{k+1}.$$

Next, we want to add and subtract terms to obtain a new expression involving the original sum. To do this we want to get the limits of summation back to $k=1$ to n. Factoring out an α and adding and subtracting terms gives

$$S_{m,n} \;=\; \alpha\left(\sum_{k=0}^{n}(k+1)^m\alpha^k-(n+1)^m\alpha^n\right) \;=\; \alpha\left(1+\sum_{k=1}^{n}(k+1)^m\alpha^k-(n+1)^m\alpha^n\right).$$

We can now use the binomial theorem to expand $(k+1)^m$ and rearrange the order of summation to get

$$S_{m,n} \;=\; \alpha\left(1+\sum_{k=1}^{n}\sum_{j=0}^{m}\binom{m}{j}k^j\alpha^k-(n+1)^m\alpha^n\right) \;=\; \alpha\left(1+\sum_{j=0}^{m}\binom{m}{j}\sum_{k=1}^{n}k^j\alpha^k-(n+1)^m\alpha^n\right).$$

The sum can be broken into the two cases, $0\le j<m$ and $j=m$. Recognizing $S_{m,n}$ and $S_{j,n}$ gives

$$S_{m,n} \;=\; \alpha\left(1+\sum_{j=0}^{m-1}\binom{m}{j}\sum_{k=1}^{n}k^j\alpha^k+\sum_{k=1}^{n}k^m\alpha^k-(n+1)^m\alpha^n\right)$$

$$= \alpha \left(1 - (n+1)^m \alpha^n + \sum_{j=0}^{m-1} \binom{m}{j} S_{j,n} + S_{m,n} \right).$$

Gathering terms and solving for $S_{m,n}$ again results in the recurrence relation (1).

Before we attack the recurrence for $S_{m,n}$, it is instructive to look at small cases:

$$\begin{aligned} S_{0,n} &= \frac{\alpha}{1-\alpha} - \alpha^n \left(\frac{\alpha}{1-\alpha} \right), \\ S_{1,n} &= \frac{\alpha}{(1-\alpha)^2} - \alpha^n \left(\frac{\alpha}{1-\alpha} n + \frac{\alpha}{(1-\alpha)^2} \right), \quad \text{and} \\ S_{2,n} &= \frac{\alpha+\alpha^2}{(1-\alpha)^3} - \alpha^n \left(\frac{\alpha}{1-\alpha} n^2 + \frac{2\alpha}{(1-\alpha)^2} n + \frac{\alpha+\alpha^2}{(1-\alpha)^3} \right). \end{aligned}$$

It appears that $S_{m,n}$ is equal to a constant with respect to n minus the product of α^n with a polynomial in n of degree m. We can easily prove this by induction using (1). If we call the first term C_m and the second term $D_{m,n}$ we can express $S_{m,n}$ as

$$S_{m,n} = C_m - D_{m,n}. \tag{2}$$

Substituting into (1), we can determine that C_m and $D_{m,n}$ are defined by the recurrences

$$C_m = \frac{\alpha}{1-\alpha} \left(1 + \sum_{j=0}^{m-1} \binom{m}{j} C_j \right) \quad \text{and} \tag{3}$$

$$D_{m,n} = \frac{\alpha}{1-\alpha} \left(\alpha^n (n+1)^m + \sum_{j=0}^{m-1} \binom{m}{j} D_{j,n} \right). \tag{4}$$

We will consider each of these recurrences in turn.

3. Solution of C_m

A good rule of thumb when solving summations and recurrences is to begin by looking at small cases. Table 1 lists the first eight values of C_m for general α. Table 2 lists the first eight values of C_m for some particular values of α. We note that the table entries for $\alpha \geq 1/2$ are integers. This observation is true only if $\alpha = (\gamma - 1)/\gamma$ for integer $\gamma \geq 2$. In this case $\alpha/(1-\alpha) = \gamma - 1$ is an integer. Using this fact and (3) we can show that C_0 is an integer, and inductively show that C_m is an integer under these conditions.

$$
\begin{aligned}
C_0 &= \frac{\alpha}{(1-\alpha)} \\
C_1 &= \frac{\alpha}{(1-\alpha)^2} \\
C_2 &= \frac{\alpha+\alpha^2}{(1-\alpha)^3} \\
C_3 &= \frac{\alpha+4\alpha^2+\alpha^3}{(1-\alpha)^4} \\
C_4 &= \frac{\alpha+11\alpha^2+11\alpha^3+\alpha^4}{(1-\alpha)^5} \\
C_5 &= \frac{\alpha+26\alpha^2+66\alpha^3+26\alpha^4+\alpha^5}{(1-\alpha)^6} \\
C_6 &= \frac{\alpha+57\alpha^2+302\alpha^3+302\alpha^4+57\alpha^5+\alpha^6}{(1-\alpha)^7} \\
C_7 &= \frac{\alpha+120\alpha^2+1191\alpha^3+2416\alpha^4+1191\alpha^5+120\alpha^6+\alpha^7}{(1-\alpha)^8}
\end{aligned}
$$

Table 1. C_m for general α.

Further examination of Table 1 reveals that the coefficient of α^j in the numerator of C_m appears to be the Euler number $\left\langle {m \atop j-1} \right\rangle$. The Euler number $\left\langle {m \atop j} \right\rangle$ is the number of permutations of $\{1, 2, \ldots, m\}$ that have j ascents [2]. Thus, we can guess that

$$C_m = \sum_{j=1}^{m} \frac{\left\langle {m \atop j-1} \right\rangle \alpha^j}{(1-\alpha)^{m+1}}. \tag{5}$$

To prove (5) we must first use induction on (3) to prove that $C_{m+1} = \alpha(dC_m/d\alpha)$. We can then use this result and induction to obtain (5) .

In an attempt to find a better expression than (5) for C_m, we will try applying exponential generating function techniques [7]. To find the exponential generating function $C(z)$, we multiply (3) by $z^m/m!$ and sum over $m \geq 0$ to obtain

$$C(z) = \sum_{m=0}^{\infty} C_m \frac{z^m}{m!} = \sum_{m=0}^{\infty} \frac{\alpha}{1-\alpha} \left(1 + \sum_{j=0}^{m-1} \binom{m}{j} C_j \right) \frac{z^m}{m!}.$$

	$\alpha=\frac{1}{5}$	$\alpha=\frac{1}{4}$	$\alpha=\frac{1}{3}$	$\alpha=\frac{1}{2}$	$\alpha=\frac{2}{3}$	$\alpha=\frac{3}{4}$	$\alpha=\frac{4}{5}$
C_0	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3	4
C_1	$\frac{5}{16}$	$\frac{4}{9}$	$\frac{3}{4}$	2	6	12	20
C_2	$\frac{15}{32}$	$\frac{20}{27}$	$\frac{3}{2}$	6	30	84	180
C_3	$\frac{115}{128}$	$\frac{44}{27}$	$\frac{33}{8}$	26	222	876	2420
C_4	$\frac{285}{128}$	$\frac{380}{81}$	15	150	2190	12180	43380
C_5	$\frac{3535}{512}$	$\frac{4108}{243}$	$\frac{273}{4}$	1082	27006	211692	972020
C_6	$\frac{26355}{1024}$	$\frac{17780}{243}$	$\frac{1491}{4}$	9366	399630	4415124	26136180
C_7	$\frac{458555}{4096}$	$\frac{269348}{729}$	$\frac{38001}{16}$	94586	6899262	107430636	819890420

Table 2. C_m for some particular values of α.

Factoring out the constant $\alpha/(1-\alpha)$ and splitting the sum into two parts gives

$$C(z) = \frac{\alpha}{1-\alpha}\left(\sum_{m=0}^{\infty}\frac{z^m}{m!} + \sum_{m=0}^{\infty}\sum_{j=0}^{m-1}\binom{m}{j}C_j\frac{z^m}{m!}\right).$$

The first sum we recognize as e^z. We can change the order of the second summation and expand the binomial coefficient to find

$$C(z) = \frac{\alpha}{1-\alpha}e^z + \frac{\alpha}{1-\alpha}\sum_{j=0}^{\infty}\sum_{m=j+1}^{\infty}\frac{C_j\, m!\, z^m}{j!\,(m-j)!\, m!}.$$

Factoring out $C_j z^j/j!$ and evaluating the summations produces

$$\begin{aligned} C(z) &= \frac{\alpha}{1-\alpha}e^z + \frac{\alpha}{1-\alpha}\sum_{j=0}^{\infty}C_j\frac{z^j}{j!}\sum_{m=j+1}^{\infty}\frac{z^{m-j}}{(m-j)!} \\ &= \frac{\alpha}{1-\alpha}e^z + \frac{\alpha}{1-\alpha}C(z)(e^z-1). \end{aligned}$$

Solving for $C(z)$ yields the closed form

$$C(z) = \frac{\alpha\, e^z}{1-\alpha e^z}. \tag{6}$$

To determine C_m, we expand $C(z)$ and find the coefficient of $z^m/m!$. Substituting the series expansion for $1/(1-\alpha e^z)$ and multiplying through the αe^z gives

$$C(z) \;=\; \alpha e^z \sum_{k=0}^{\infty} \alpha^k e^{zk} \;=\; \sum_{k=0}^{\infty} \alpha^{k+1} e^{z(k+1)}.$$

Next we expand $e^{z(k+1)}$ and change the order of summation to find the coefficient of $z^m/m!$,

$$C(z) \;=\; \sum_{k=0}^{\infty} \alpha^{k+1} \sum_{m=0}^{\infty} \frac{z^m (k+1)^m}{m!} \;=\; \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \alpha^{k+1}(k+1)^m \frac{z^m}{m!}.$$

Thus, C_m is given by

$$C_m \;=\; \sum_{k=0}^{\infty} \alpha^{k+1}(k+1)^m \;=\; \sum_{k=1}^{\infty} \alpha^k k^m. \tag{7}$$

We can immediately recognize that $C_m = \lim_{n\to\infty} S_{m,n}$. It follows from (7) and (2) that

$$D_{m,n} \;=\; C_m - S_{m,n} \;=\; \sum_{k=n+1}^{\infty} \alpha^k k^m, \tag{8}$$

which tends to 0 as $n \to \infty$.

We can now use (7) to provide an intuitive proof of (5). The identity

$$x^m = \sum_{j=0}^{m-1} \left\langle {m \atop j} \right\rangle \binom{x+j}{m}, \quad \text{integer } m \geq 0,$$

uses Euler numbers to relate the binomial coefficients to polynomial powers [2]. Letting $x = k$, substituting into (7), and changing the order of summation gives

$$C_m \;=\; \sum_{k=1}^{\infty} \alpha^k k^m \;=\; \sum_{k=1}^{\infty} \alpha^k \sum_{j=0}^{m-1} \left\langle {m \atop j} \right\rangle \binom{k+j}{m} \;=\; \sum_{j=0}^{m-1} \left\langle {m \atop j} \right\rangle \sum_{k=0}^{\infty} \binom{k+j+1}{m} \alpha^k.$$

The inner sum is the generating function for the sequence $\binom{k+j+1}{m}$ and converges to $\alpha^{j+1}/(1-\alpha)^{m+1}$ for $0 < |\alpha| < 1$. Substituting this value yields the desired result,

$$C_m \;=\; \sum_{j=0}^{m-1} \left\langle {m \atop j} \right\rangle \frac{\alpha^{j+1}}{(1-\alpha)^{m+1}} \;=\; \sum_{j=1}^{m} \frac{\left\langle {m \atop j-1} \right\rangle \alpha^j}{(1-\alpha)^{m+1}}.$$

	$\alpha = .2$		$\alpha = .5$		$\alpha = .8$	
	Actual	Approximate	Actual	Approximate	Actual	Approximate
C_0	0.2500	0.6213	1.0000	1.4427	4.0000	4.4814
C_1	0.3125	0.3861	2.0000	2.0814	20.0000	20.0831
C_2	0.4688	0.4797	6.0000	6.0056	180.0000	180.0019
C_3	0.8984	0.8942	26.0000	25.9926	2420.0000	2419.9918
C_4	2.2266	2.2225	150.0000	149.9975	43380.0000	43379.9991
C_5	6.9043	6.9046	1082.0000	1082.0030	972020.0000	972020.0039
C_6	25.7373	25.7403	9366.0000	9366.0025	26136180.0000	26136180.0009
C_7	111.9519	111.9533	94586.0000	94585.9975	819890420.0000	819890419.9960

Table 3. Approximate values of C_m for particular α.

Thus we find that the infinite series defined by C_m converges for all $0 < |\alpha| < 1$.

A closed form solution for (7) can be found for any particular m through the technique of repeated differentiation that we used in Section 2. However, there is no known closed form solution for general m. We can find a good asymptotic expression for C_m by integrating $\alpha^x x^m dx$ over $x \geq 0$,

$$C_m \sim \int_0^\infty \alpha^x x^m dx = \int_0^\infty e^{-x \ln \frac{1}{\alpha}} x^m dx = \frac{\Gamma(m+1)}{(\ln \frac{1}{\alpha})^{m+1}} = \frac{m!}{(\ln \frac{1}{\alpha})^{m+1}}. \tag{9}$$

This integral exists if $\ln(1/\alpha) > 0$, which implies that $0 < \alpha < 1$. Table 3 shows the first eight values of C_m and the asymptotic for some particular values of α.

The special case when $\alpha = 1/2$ is of interest. The sequence

$$F_m = 1, 3, 13, 75, 541, 4683, 47293, 545835, \ldots$$

which is defined by $F_m = C_m/2$ for $m \geq 1$, is the number of preferential arrangements of m distinguishable items [3, 5].

m	$i=7$	$i=6$	$i=5$	$i=4$	$i=3$	$i=2$	$i=1$	$i=0$
0								1
1							1	2
2						1	4	6
3					1	6	18	26
4				1	8	36	104	150
5			1	10	60	260	750	1082
6		1	12	90	520	2250	6492	9366
7	1	14	126	910	5250	22722	65562	94586

Table 4. Some values of $d_{i,j}$ for $\alpha = 1/2$.

4. Solution of $D_{m,n}$

As shown in Section 2, $D_{m,n}$ can be expressed as the product of α^n with a polynomial in n of degree m,

$$D_{m,n} = \alpha^n \sum_{i=0}^{m} d_{m,i} n^i. \tag{10}$$

Noting that $d_{m,i} = 0$ for $i > m$, applying (4), and equating coefficients of n^i yields

$$d_{m,i} = \frac{\alpha}{1-\alpha} \left(\binom{m}{i} + \sum_{j=i}^{m-1} \binom{m}{j} d_{j,i} \right). \tag{11}$$

As we did with C_m, it is instructive to look at small cases to try to gain insight into the behavior of the coefficients $d_{m,i}$. Since it is difficult to compute $d_{m,i}$ for a general α, we will look at the specific case $\alpha = 1/2$. Table 4 shows the values of $d_{m,i}$ for $m \le 7$.

The most obvious result is that $d_{m,0} = C_m$. This can easily be proved by induction using (11) and (3). Less obvious is the result that

$$d_{m,i} = \binom{m}{i} C_{m-i}. \tag{12}$$

This result can be seen by noting that $d_{m,i} = (m/i) d_{m-1,i-1}$. Working back along the diagonal gives

$$d_{m,i} = \frac{m}{i}\frac{m-1}{i-1}\cdots\frac{m-i+1}{1} d_{m-i,0} = \binom{m}{i} C_{m-i}.$$

Checking the small cases for $S_{m,n}$ considered in Section 2, it can be seen that (12) holds for general α when

$m \geq 1$. We can use induction to prove this for all $m \geq 1$. The proof involves showing that if (12) is true for the first m diagonals in the table, then it is true for the $(m+1)$th diagonal. The coefficients $d_{m+i,i}$, for $i \geq 0$, define the mth diagonal in the table. The base step is $d_{i,i} = \alpha/(1-\alpha) = C_0$. The inductive step assumes (12) is true for the first m_0 diagonals and then uses (11), algebraic manipulation, and (3) to prove that $d_{m_0+1+i,i} = \binom{m_0+1+i}{i} C_{m_0+1}$. Application of (10) and (12) then yields

$$D_{m,n} = \alpha^n \sum_{k=0}^{m} \binom{m}{k} C_{m-k} n^k, \quad 0 < |\alpha| < 1. \tag{13}$$

We could attempt to find a closed form expression for $D_{m,n}$ by substituting (7) into (13). However, this approach yields the result found in (8). Alternatively, we can attack $D_{m,n}$ with generating functions as we did with C_m. First, we find the exponential generating function, $D_n(z)$. Multiplying $D_{m,n}$ by $z^m/m!$, summing over m, and substituting (4) gives

$$D_n(z) = \sum_{m=0}^{\infty} D_{m,n} \frac{z^m}{m!} = \sum_{m=0}^{\infty} \frac{\alpha}{1-\alpha} \left(\alpha^n (n+1)^m + \sum_{j=0}^{m-1} \binom{m}{j} D_{j,n} \right) \frac{z^m}{m!}.$$

Breaking the sum into two parts, recognizing $e^{(n+1)z}$, rearranging the order of the second sum, and expanding the binomial coefficient yields

$$\begin{aligned} D_n(z) &= \frac{\alpha^{n+1}}{1-\alpha} \sum_{m=0}^{\infty} (n+1)^m \frac{z^m}{m!} + \frac{\alpha}{1-\alpha} \sum_{m=0}^{\infty} \sum_{j=0}^{m-1} \binom{m}{j} D_{j,n} \frac{z^m}{m!} \\ &= \frac{\alpha^{n+1}}{1-\alpha} e^{(n+1)z} + \frac{\alpha}{1-\alpha} \sum_{j=0}^{\infty} \sum_{m=j}^{\infty} \frac{D_{j,n} z^m}{j!\,(m-j)!}. \end{aligned}$$

$D_{j,n} z^j / j!$ can be factored out and the two sums evaluated to get an expression involving $D_n(z)$,

$$D_n(z) = \frac{\alpha^{n+1}}{1-\alpha} e^{(n+1)z} + \frac{\alpha}{1-\alpha} \sum_{j=0}^{\infty} D_{j,n} \frac{z^j}{j!} \sum_{m=j}^{\infty} \frac{z^{m-j}}{(m-j)!} = \frac{\alpha^{n+1}}{1-\alpha} e^{(n+1)z} + \frac{\alpha}{1-\alpha} D_n(z)(e^z - 1).$$

Gathering terms and solving for $D_n(z)$ gives

$$D_n(z) = \alpha^n e^{nz} \frac{\alpha e^z}{1 - \alpha e^z}. \tag{14}$$

Now we can expand $D_n(z)$ to find $D_{m,n}$, the coefficient of $z^m/m!$. We recognize that $D_n(z) = \alpha^n e^{nz} C(z)$ is the binomial convolution of the two exponential generating functions

$$e^{nz} = \sum_{m=0}^{\infty} n^m \frac{z^m}{m!} \quad \text{and} \quad C(z) = \sum_{m=0}^{\infty} C_m \frac{z^m}{m!}.$$

Thus

$$D_n(z) = \sum_{m=0}^{\infty} \alpha^n \sum_{k=0}^{m} \binom{m}{k} n^k C_{m-k} \frac{z^m}{m!},$$

and we find that the coefficient of $z^m/m!$ is given by (13).

We would also like to find an asymptotic expression for $D_{m,n}$. Unfortunately, we are unable to use integration to approximate $D_{m,n}$ from (8) as we did to approximate C_m from (7). However, we can apply techniques from complex analysis to the generating function $D_n(z)$ to find an asymptotic for $D_{m,n}$.

The value we want to find, $D_{m,n}$, is the coefficient of $z^m/m!$ in (14). Because $D_n(z)$ is an exponential generating function, we have $D_{m,n} = D_n^{(m)}(0)$, the mth derivative of $D_n(z)$ evaluated at $z = 0$. Since $D_n(z)$ is analytic in a small positively oriented circle Γ with the origin in its interior, Cauchy's integral coefficient formula gives

$$D_{m,n} = D_n^{(m)}(0) = \frac{m!}{2\pi i} \oint_{\Gamma} \frac{D_n(z)}{z^{m+1}}\, dz = \frac{m!}{2\pi i} \oint_{\Gamma} \frac{\alpha^{n+1} e^{(n+1)z}}{z^{m+1}(1 - \alpha e^z)}\, dz. \tag{15}$$

The function $D_n(z)$ has singularities at $z_k = \ln(1/\alpha) + 2\pi k i$ for $k \in \mathcal{Z}$. Using the contour of integration $\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$ as shown in Figure 1, Cauchy's residue theorem gives

$$\frac{1}{2i\pi} \oint_{\varphi} \frac{D_n(z)}{z^{m+1}}\, dz = -\sum_{k=-j}^{j} \text{Residue}\big(D_n(z)/z^{m+1}, z = z_k\big), \tag{16}$$

where Residue$(f(z), z = x)$ defines the residue of $f(z)$ at $z = x$. The integral in (16) along φ_1 is the integral we want to evaluate in (15). The integrals along φ_2 and φ_4 cancel each other and the integral along φ_3 tends to 0 as $j \to \infty$. Thus we get

$$D_{m,n} = \frac{m!}{2\pi i} \oint_{\Gamma} \frac{D_n(z)}{z^{m+1}}\, dz = -m! \sum_{k \in \mathcal{Z}} \text{Residue}\big(D_n(z)/z^{m+1}, z = z_k\big).$$

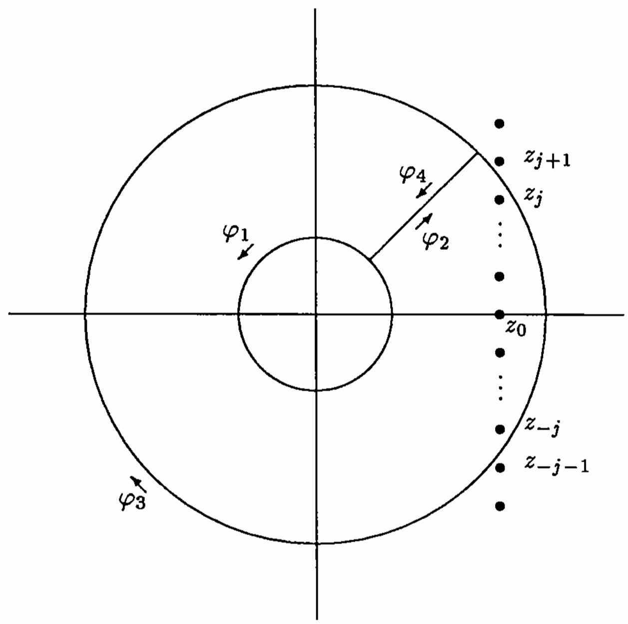

Figure 1. Contour of integration $\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$ for $D_n(z)/z^{m+1}$.

The residue of $D_n(z)/z^{m+1}$ at each singularity z_k is $-z_k^{-(m+1)}$, which gives

$$D_{m,n} \;=\; -m! \sum_{k\in\mathcal{Z}} \left(-z_k^{-(m+1)}\right) \;=\; \frac{m!}{\left(\ln\frac{1}{\alpha}\right)^{m+1}} \sum_{k\in\mathcal{Z}} \left(1+\frac{2\pi k i}{\ln\frac{1}{\alpha}}\right)^{-(m+1)}.$$

The dominant term corresponds to $k = 0$, so we get

$$D_{m,n} \;\sim\; \frac{m!}{\left(\ln\frac{1}{\alpha}\right)^{m+1}}. \tag{17}$$

5. Value of $S_{n,m}$

Using (5) and (13) we find that for $0 < |\alpha| < 1$,

$$S_{m,n} \;=\; \sum_{j=1}^{m} \frac{\left\langle {m \atop j-1} \right\rangle \alpha^j}{(1-\alpha)^{m+1}} - \alpha^n \sum_{i=0}^{m} \binom{m}{i} C_{m-i}$$

$$= \sum_{j=1}^{m} \frac{\left\langle {m \atop j-1} \right\rangle \alpha^j}{(1-\alpha)^{m+1}} - \alpha^n \sum_{i=0}^{m} \binom{m}{i} \sum_{j=1}^{m-i} \frac{\left\langle {m-i \atop j-1} \right\rangle \alpha^j}{(1-\alpha)^{m+1-i}}. \tag{18}$$

For $\alpha > 1$, the infinite series for C_m and $D_{m,n}$ grow without bounds and this result does not apply.

Examination of the small cases in Section 2 reveals that $S_{m,n}$ can be obtained from (18) for small m. Using the fact that $S_{m+1,n} = \alpha(dS_{m,n}/d\alpha)$ and induction on m, it can be proved that (18) holds for any $\alpha \neq 1$. It can also be shown that the singularity at $\alpha = 1$ is removable. Application of the appropriate limit theorems will then lead to expressions for the sum of the powers of integers.

We would like to find an asymptotic value for $S_{m,n}$ for $0 < |\alpha| < 1$. Unfortunately, none of the methods we have examined so far will produce such an expression. The integration method that we used for finding (9) will not work. Even the complex analysis technique we used to find (17) does not yield an expression for $S_{m,n}$. The generating function $S_n(z) = C(z) - D_n(z)$ has a removable singularity. Thus it has no poles and we cannot apply the residue theorem as we did in (16) for $D_m(z)$.

If we consider the sequence $\langle k^m \alpha^k \rangle$ we notice that as $k \to \infty$, k^m grows large and α^k grows exponentially small. The individual discrete terms $\alpha^k k^m$ in $S_{m,n}$ are maximized when $k^m \alpha^k = (k+1)^m \alpha^{k+1}$. The value of k that maximizes this expression is

$$k \approx \frac{m}{\ln \frac{1}{\alpha}} + \frac{1}{2} + O\left(\frac{1}{m}\right).$$

Values of n less than, greater than, or equal to this value of k will produce different asymptotics for $S_{m,n}$. Thus, we will consider the behavior of $S_{m,n}$ as $m, n \to \infty$ when $n = \beta m$ where $\beta < 1/\ln(1/\alpha)$, $n = m/\ln(1/\alpha) + \gamma\sqrt{m}$, and $n = \beta m$ where $\beta > 1/\ln(1/\alpha)$.

Case 1. The growth rate is $n = \beta m$, where $\beta < 1/\ln(1/\alpha)$. The individual terms in $S_{m,n}$ are exponentially increasing. The last term $\alpha^n n^m$ is therefore dominant. Factoring the dominant term out of the summation gives

$$S_{m,n} = \sum_{k=1}^{n} \alpha^k k^m = \alpha^n n^m \sum_{k=0}^{n-1} \frac{1}{\alpha^k} \left(1 - \frac{k}{n}\right)^m.$$

We can express the $(1-k/n)^m$ as a power of e and use power series manipulation to get

$$\left(1-\frac{k}{n}\right)^m = \exp\left(m\ln\left(1-\frac{k}{n}\right)\right) = \exp\left(m\left(-\frac{k}{n}+O\left(\frac{k^3}{n^3}\right)\right)\right) = e^{-km/n}\exp\left(O\left(\frac{mk^2}{n^2}\right)\right).$$

Since $n = O(m)$, the power series expansion of the exponential gives $\left(1-\frac{k}{n}\right)^m = e^{-km/n}\left(1+O\left(\frac{k^2}{m}\right)\right)$. Substituting this into the expression for $S_{m,n}$ and approximating the summation by the infinite series yields

$$S_{m,n} = \alpha^n n^m \sum_{k=0}^{n-1}\frac{1}{\alpha^k}e^{-km/n}\left(1+O\left(\frac{k^2}{m}\right)\right) \sim \alpha^{\beta m}(\beta m)^m\left(1-\frac{1}{\alpha}e^{-1/\beta}\right)^{-1}\left(1+O\left(\frac{1}{m}\right)\right).$$

Case 2. The growth rate is $n = m/\ln(1/\alpha)+\gamma\sqrt{m}$. Using the commutative law to replace k with $n+1-k$, letting $N = m/\ln(1/\alpha)$, and shifting the limits of summation gives

$$\begin{aligned} S_{m,n} &= \sum_{k=1}^{n}\alpha^k k^m = \sum_{k=1}^{n}\alpha^{n+1-k}(n+1-k)^m \\ &= \sum_{k=1}^{N+\gamma\sqrt{m}}\alpha^{N+\gamma\sqrt{m}+1-k}(N+\gamma\sqrt{m}+1-k)^m \\ &= \alpha^N N^m \sum_{k=-\gamma\sqrt{m}}^{N-1}\frac{1}{\alpha^k}\left(1-\frac{k}{N}\right)^m. \end{aligned}$$

Using power series to expand $(1-k/N)^m$ and substituting for N, we find that

$$S_{m,n} = \frac{\left(\frac{m}{e}\right)^m}{\left(\ln\frac{1}{\alpha}\right)^m}\sum_{k=-\gamma\sqrt{m}}^{N-1}\exp\left(-\frac{k^2\ln^2\alpha}{2m}\right)\left(1+O\left(\frac{k^3}{m^2}\right)\right).$$

By Euler's summation formula, we have

$$\begin{aligned} \sum_{k=-\gamma\sqrt{m}}^{N-1}\exp\left(-\frac{k^2\ln^2\alpha}{2m}\right)\left(1+O\left(\frac{k^3}{m^2}\right)\right) &= \frac{\sqrt{m}}{\ln\frac{1}{\alpha}}\int_{\gamma\ln\alpha}^{\sqrt{m}}\exp\left(-\frac{y^2}{2}\right)dy\left(1+O\left(\frac{1}{\sqrt{m}}\right)\right) \\ &= \frac{\sqrt{2\pi m}}{\ln\frac{1}{\alpha}}\left(\frac{1}{2}+\mathcal{N}\left(\gamma\ln\frac{1}{\alpha}\right)\right)\left(1+O\left(\frac{1}{\sqrt{m}}\right)\right), \end{aligned}$$

where

$$\mathcal{N}(t) = \int_0^t \exp\left(-\frac{y^2}{2}\right)dy.$$

Therefore,

$$S_{m,n} = \frac{\sqrt{2\pi m}\left(\frac{m}{e}\right)^m}{\left(\ln\frac{1}{\alpha}\right)^{m+1}}\left(\frac{1}{2}+\mathcal{N}\left(\gamma\ln\frac{1}{\alpha}\right)\right)\left(1+O\left(\frac{1}{\sqrt{m}}\right)\right).$$

Recognizing the asymptotic for $m!$ gives

$$S_{m,n} = \frac{m!}{\left(\ln\frac{1}{\alpha}\right)^{m+1}}\left(\frac{1}{2}+\mathcal{N}\left(\gamma\ln\frac{1}{\alpha}\right)\right)\left(1+O\left(\frac{1}{\sqrt{m}}\right)\right).$$

Case 3. The growth rate is $n=\beta m$, where $\beta > 1/\ln(1/\alpha)$. By the same derivation as in Case 2, we have

$$\begin{aligned} S_{m,n} &= \sum_{k=1}^{n}\alpha^k k^m = \alpha^N N^m \frac{\sqrt{2\pi m}}{\ln\frac{1}{\alpha}}\left(\frac{1}{2}+\mathcal{N}\left(\beta\sqrt{m}\ln\frac{1}{\alpha}\right)\right)\left(1+O\left(\frac{1}{\sqrt{m}}\right)\right) \\ &= \alpha^N N^m \frac{\sqrt{2\pi m}}{\ln\frac{1}{\alpha}}\left(1+O\left(\frac{1}{\sqrt{m}}\right)\right) \\ &= \frac{\sqrt{2\pi m}\left(\frac{m}{e}\right)^m}{\left(\ln\frac{1}{\alpha}\right)^{m+1}}\left(1+O\left(\frac{1}{\sqrt{m}}\right)\right) \\ &= \frac{m!}{\left(\ln\frac{1}{\alpha}\right)^{m+1}}\left(1+O\left(\frac{1}{\sqrt{m}}\right)\right). \end{aligned}$$

A more exact analysis can be obtained from (2). Using complex analysis techniques we can show that

$$C_m = \frac{m!}{\left(\ln\frac{1}{\alpha}\right)^{m+1}}\sum_{k\in\mathcal{Z}}\left(1+\frac{2\pi k i}{\ln\frac{1}{\alpha}}\right)^{-m-1}.$$

The imaginary part of the summation cancels out so we get the exact expression

$$C_m = \frac{m!}{\left(\ln\frac{1}{\alpha}\right)^{m+1}}\sum_{k\in\mathcal{Z}}\Re\left(1+\frac{2\pi k i}{\ln\frac{1}{\alpha}}\right)^{-m-1}.$$

The terms in $D_{m,n}$ decrease exponentially, so its value is bounded by $O\big(\alpha^n n^m\big) = O\big(C_m\alpha^{(\beta-1/\ln(1/\alpha))m}\big)$.

Hence we have

$$S_{m,n} = C_m - O\left(C_m\alpha^{(\beta-1/\ln(1/\alpha))m}\right) = \frac{m!}{\left(\ln\frac{1}{\alpha}\right)^{m+1}}\sum_{k\in\mathcal{Z}}\Re\left(1+\frac{2\pi k i}{\ln\frac{1}{\alpha}}\right)^{-m-1} O\left(1+\alpha^{(\beta-1/\ln(1/\alpha))m}\right).$$

6. Conclusion

Evaluating the summation $S_{m,n} = \sum_{k=1}^{n} \alpha^k k^m$ illustrates many techniques used in the mathematical analysis of algorithms and, as such, it provides a good case study for solving summations and recurrences.

Several methods were employed to evaluate the sum. Direct calculation produced values for small m but no general solution. Both transformation techniques and perturbation yielded a recurrence for $S_{m,n}$ which we were able to express as the difference of two other recurrences, C_m and $D_{m,n}$. Solutions were found for each recurrence using induction proofs and also by using generating function techniques. While no general closed form was found for $S_{m,n}$, a closed form expression valid for every value of m was found. Various methods were employed to find asymptotics for C_m, $D_{m,n}$, and $S_{m,n}$.

The analysis also produced two interesting results about the infinite series $\sum_{k=0}^{\infty} \alpha^k k^m$. The value of the series was defined by a recurrence relation as a function of m. The series was also shown to have a closed form solution involving the Euler numbers. Either representation can be used to evaluate the series for particular values of α and m.

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