Asymptotic Notations.

Definition 2: Let $T(n)$ be a function.

(Asymptotic Lower Bounds) Given another function $g(n)$, we say $T(n) = \Omega(g(n))$ iff $\exists$ constants $\varepsilon > 0$ and n_0 s.t

$$\forall n \geq n_0. \quad T(n) \geq \varepsilon \cdot g(n).$$

[Recall. $T(n) = O(f(n))$ "$T(n) \leq c \cdot f(n)$"]

Definition 3: If $T(n) = O(f(n))$ and $T(n) = \Omega(f(n))$,

(Asymptotic Tight Bounds) we say that $T(n) = \Theta(f(n))$

Examples/Exercises:

Q1: $5n = O(n^2)$? True. ($c = 5$ $n_0 = 1$. $\forall n \geq n_0$ $5n \leq 5n^2 = c \cdot n^2$)

Q2: $5n = \Theta(n^2)$? False. ($5n \neq \Omega(n^2)$. Fix n_0, ε. Pick $n = \max(n_0, \frac{5}{\varepsilon}) + 1$.)

Q3: $\log_2 n = O(n^{0.01})$? True. (Statement (2.8) on page 41 of KT book).

Q4: $(1.01)^n = \Omega(n)$? True. (Statement (2.9) on page 42 of KT book)

Properties of O, Ω, Θ:

Transitivity: ① If $f = O(g)$ and $g = O(h)$, then $f = O(h)$.

Proof: $\forall n \geq n_0$ $f(n) \leq c \cdot g(n)$.

$\forall n \geq n_0'$ $g(n) \leq c' \cdot h(n)$

[$\forall n \geq \underbrace{\max(n_0, n_0')}_{\text{new } n_0}$ $f(n) \leq \underbrace{(c \cdot c')}_{\text{new } c} \cdot h(n)$]

② If $f = \Omega(g) \wedge g = \Omega(h)$, then $f = \Omega(h)$

③ If $f = \Theta(g) \wedge g = \Theta(h)$, then $f = \Theta(h)$.

Sum of Functions: ④ If $f = O(h)$ and $g = O(h)$, then $f + g = O(h)$.

⑤ If $g(n) \geq 0$ and $g = O(f)$, then $f + g = \Theta(f)$.

Proof sketch: When $n \to +\infty$, $f \leq f + g \leq (c+1) \cdot f$.

this is the c used in $g = O(f)$

Properties of O, Ω, Θ.

⑥ If $f = \Theta(g)$, then $g = \Theta(f)$.

Proof:

$f = \Theta(g)$ $\Updownarrow$

$\exists c, \varepsilon, n_0 \ \forall n \geq n_0$

$\boxed{\varepsilon \cdot g(n) \leq f(n)} \leq c \cdot g(n)$

$\Downarrow$

$\varepsilon' = \frac{1}{c}$ ... $c' = \frac{1}{\varepsilon}$

$\Downarrow$

$\exists c', \varepsilon', n_0' \ \forall n \geq n_0$

$\varepsilon' \cdot f(n) \leq g(n) \leq c' \cdot f(n)$

$\Updownarrow$

$g = \Theta(f)$

When $n \geq \ldots$

$\varepsilon \cdot g \leq f \leq c \cdot g \iff \frac{1}{c} \cdot f \leq g \leq \frac{1}{\varepsilon} \cdot f$

⑦ If $f = O(g)$ then $g = \Omega(f)$.

$$f = O(\log_2 n) \iff f = O(\log_{10} n)$$

$$\log_a n = \frac{\ln n}{\ln a} \qquad \log_b n = \frac{\ln n}{\ln b}.$$

$$\frac{\log_{10} n}{\log_2 n} = \frac{\frac{\ln n}{\ln 10}}{\frac{\ln n}{\ln 2}} = \frac{\ln 2}{\ln 10} = \text{constant}.$$

This is why we often write O

$$f = O(g) \iff \exists c. \quad \lim_{n \to +\infty} \frac{f(n)}{g(n)} \leq c$$

$$f = \Omega(g) \iff \exists \varepsilon \quad \lim_{n \to +\infty} \frac{f(n)}{g(n)} \geq \varepsilon.$$