Maximum Flow

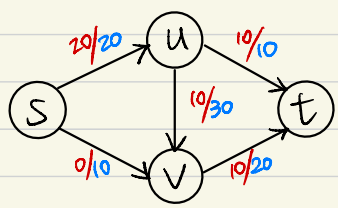
Input: a directed graph $G = (V, E, c)$
where each edge $e \in E$ has capacity $c_e > 0$.
a source node $s \in V$.
a sink node $t \in V$.

Output: A maximum s-t flow of G.

Definition (s-t flow): A flow f on a graph $G = (V, E, c)$ associate a real number $f_e \geq 0$ to each edge e such that:

① $0 \leq f_e \leq c_e$ (capacity constraint).
② $\forall v \neq s, t$, flow into v = flow out of v.
(conservation / Kirchhoff's law)

Example. $G =$
$c =$
$f =$



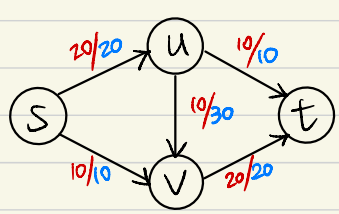
capacity
flow.

The value of an s-t flow
= (total flow out of s) − (total flow into s)
= (total flow into t) − (total flow out of t).

$v(f) = 20$ in the example above

Maximum s-t flow: an s-t flow with maximum value.

$G =$
$c =$
$f =$



$v(f) = 30$.

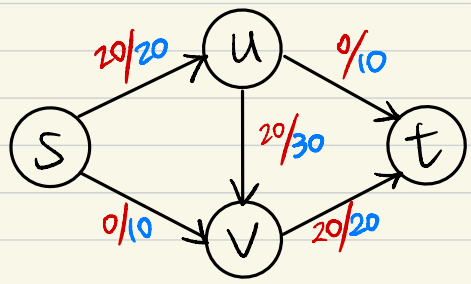
This is the maximum s-t flow on G.
(because capacity leaving s is all used up)

Computing Maximum Flow

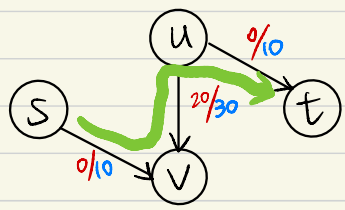
Ford-Fulkerson Algorithm [1956]:

- find a path from s to t where every edge on this path has unused capacity.
- Send flow along this path and repeat.

However:



This is not a maximum flow.
If we look at edges that are not "used up":



There is no path from s to t!

Q: How to fix this?
A: Send 10 units of flow on the path $s \to v \to u \to t$ where

Sending flow from v to u $\iff$ "undo" some flow from u to v.

Residue Graph

Given a graph $G=(V,E,c)$ and a flow f on G.

the residue graph G_f is defined as.

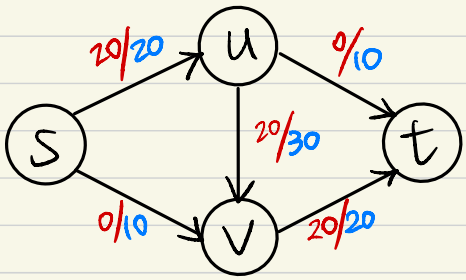
1) G_f is on V
2) For every edge $e=(u,v)$ in G:

   2a) (forward edges) if $f_e < c_e$, we add (u,v) to G_f with capacity $(c_e - f_e)$.

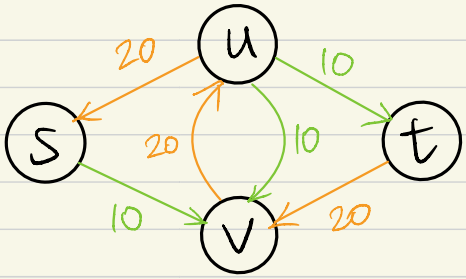
   2b) (backward edges) if $f_e > 0$, we add (v,u) to G_f with capacity f_e.

Example 1

$G, c, f =$



$G_f =$



forward edge

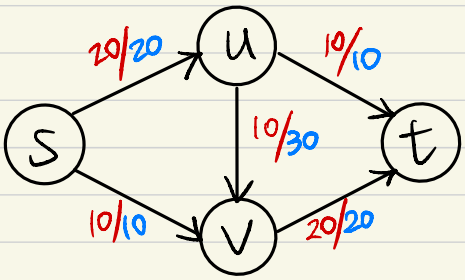
backward edge

Now there is an s-t path in G_f!

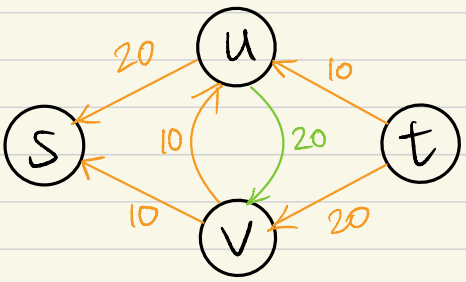
$s \to v \to u \to t$

Example 2

$G, c, f =$



$G_f =$



forward edge

backward edge

Augmenting Path. a simple path P from s to t in G_f.

Ford-Fulkerson Algorithm:

$f \leftarrow 0$

WHILE (there is an augmenting path P in G_f)

  augment(f, P)

Augment(f, P):

$b \leftarrow$ the minimum capacity of edges in P in the residue graph G_f.

FOR every $e \in P$:

  if e is a forward edge: $f_e \leftarrow f_e + b$

  if e is a backward edge: $f_e \leftarrow f_e - b$.

We can show that

1. The new f has larger value
2. The new f is a valid s-t flow

(see page 343 of the textbook)

respects capacity

satisfies conservation