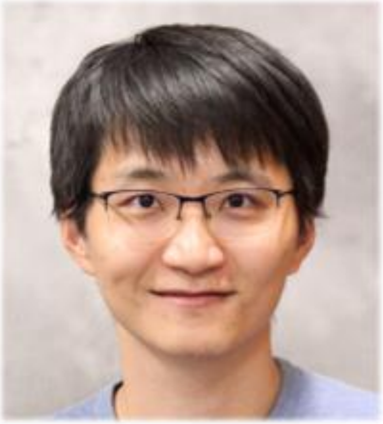
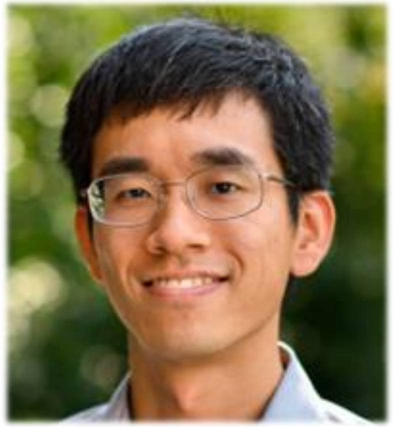
On the Communication Complexity of Maximum Matching and Shortest Paths



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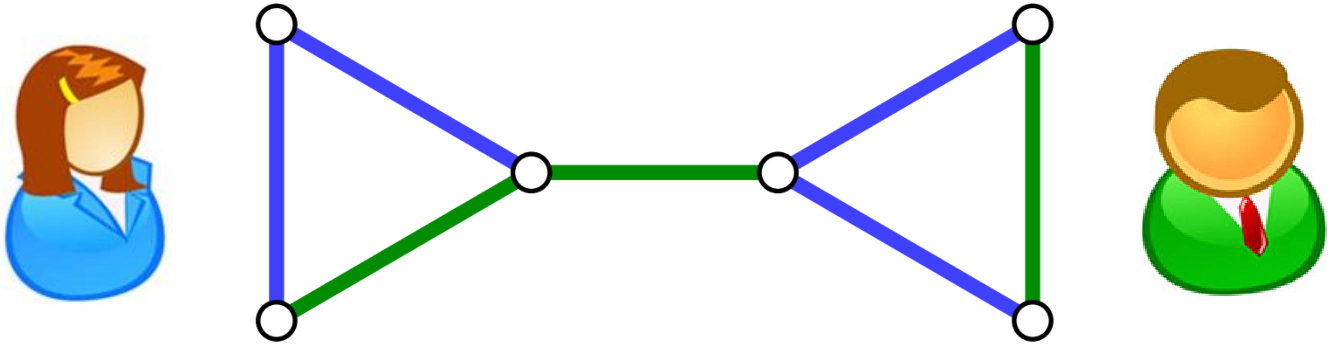
Communication Complexity of Graph Problems



- Input: $G = (V, E)$, where $E = E_A \cup E_B$.
  Alice knows (V, E_A) and Bob knows (V, E_B).
- Goal: solve a problem on G while minimizing communication.

Prior Work [BBEMN'22]

[BBEMN'22] gave protocols for several graph problems, including:

- $\tilde{O}(n)$ bits for bipartite matching. $n = |V|$
- $\tilde{O}(n)$ bits for negative cycle detection and negative-weight SSSP.

(assuming $\mathrm{poly}(n)$ edge weights)

Motivating questions:

- General matching?
- Simpler and more direct protocols?

Our Results

- $\tilde{O}(n^{1.5})$ bits for general matching.
  - Same bound known by simulating blossom algorithms [MV'80, G'17]; we give a more direct protocol.
- $\tilde{O}(n)$ bits for negative cycle detection and negative-weight SSSP.
  - Replaces a chain of reductions in [BBEMN'22] with a more direct protocol.
- A combinatorial $\tilde{O}(n)$-bit protocol for bipartite matching.

Outline

- $\tilde{O}(n^{1.5})$ bits for general matching
- $\tilde{O}(n)$ bits for negative cycle detection and negative SSSP

$\tilde{O}(n^{1.5})$ Protocol for General Matching

It suffices to decide whether G has a perfect matching (PM).

- Phase 1: $\tilde{O}(n^2/k)$ communication
  - Find a matching of size $\frac{n}{2} - k$, or certify that no PM exists.
- Phase 2: $\tilde{O}(kn)$ communication
  - Simulate at most k iterations of Blossom.

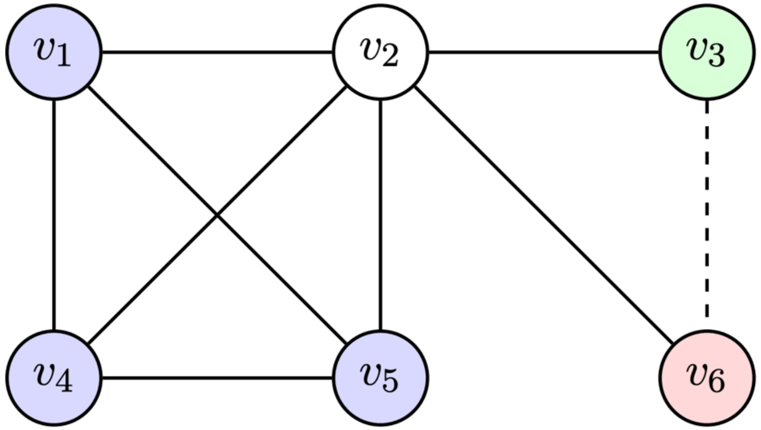
Choosing $k = \sqrt{n}$ gives $\tilde{O}(n^{1.5})$ communication.

Tutte-Berge Covers

Definition: $C = (A, S_1, \ldots, S_t)$, where $(A, S_1, \ldots, S_t)$ is a partition of V and each $|S_i|$ is odd.

C covers an edge e if

- one endpoint of e is in A, or
- both endpoints of e are in the same S_i.



Example: $A = \{v_2\}$, $S_1 = \{v_1, v_4, v_5\}$, $S_2 = \{v_3\}$, $S_3 = \{v_6\}$.

Value of Tutte-Berge Covers

Define the value of a cover C as $\delta(C) = |A| + \Sigma_i \frac{|S_i| - 1}{2}$.

C covers at most $\delta(C)$ edges of any matching:

- at most $|A|$ matching edges incident to A, and
- at most $\frac{|S_i| - 1}{2}$ matching edges inside each S_i, since $|S_i|$ is odd.

Tutte-Berge formula: size of maximum matching $=$ value of minimum cover.

Idea: no low-value covers $\Longrightarrow$ large matching.

Iteratively reveal an edge that invalidates the most low-value covers.

Eliminating Tutte-Berge Covers

Let H be the revealed graph. Let $\mathcal{C} := \left\{\text{covers } C \text{ of } H : \delta(C) = \frac{n}{2} - k\right\}$.

Initially, $H_0 = (V, \emptyset)$ and $|\mathcal{C}_0| \leq (n+1)^n$.

If G has a perfect matching M, then each $C \in \mathcal{C}$ misses at least k edges of M

$\Longrightarrow$ some $e \in M$ invalidates at least $\frac{k}{|M|} = \frac{2k}{n}$ -fraction of $\mathcal{C}$

$\Longrightarrow$ $|\mathcal{C}_T| \leq \left(1 - \frac{2k}{n}\right)^T |\mathcal{C}_0|$ $\Longrightarrow$ $T = O(n^2 \log n / k)$ eliminates all covers.

Outline

- $\tilde{O}(n^{1.5})$ bits for general matching
- $\tilde{O}(n)$ bits for negative cycle detection and negative SSSP

A Cutting-Plane Framework

Goal: decide feasibility of a polytope P for some graph problem.

- Example [BBEMN'22]: $P = \left\{\text{fractional vertex covers of size } \frac{n}{2} - 1\right\}$.

Cutting-plane protocol:

1. Maintain a public convex body K.
2. Alice and Bob locally check if the centroid of K violates one of their edge constraints. If yes, communicate a violated constraint and cut K, and repeat Step 2.
3. Eventually either find a feasible point, or certify that P has very small volume.

Negative Cycle Detection

Input: directed graph $G = (V, E, w)$ with integer weights $w(e) \in [-W, W]$.

We consider $P = \left\{ p \in [0, nW]^n : p_v - p_u \leq w(u, v) + \frac{1}{n^2},\ \forall (u, v) \in E \right\}$.

- If G has a negative cycle, then $P = \emptyset$:
  - Let $C = (v_1, \ldots, v_k)$ be a negative cycle.

    $0 = \Sigma_{i=1}^{k} \left(p_{v_{i+1}} - p_{v_i} \right) \leq \Sigma_{i=1}^{k} w(v_i, v_{i+1}) + \frac{k}{n^2} \leq -1 + \frac{k}{n^2} < 0$ gives a contradiction.
- If G has no negative cycles, then $\mathrm{vol}(P) \geq n^{-2n}$.

Negative-Weight SSSP

We use a reweighting idea from Johnson's algorithm:

For a vertex potential $\phi \in \mathbb{R}^n$, define $w_\phi(u, v) = w(u, v) + \phi(u) - \phi(v)$.

The length of every s-to-t path shifts by $\phi(s) - \phi(t)$, so shortest paths are preserved.

If G has no negative cycles, the previous protocol returns a feasible point $p \in P$ such that

$$p_v - p_u \leq w(u, v) + \frac{1}{n^2} \implies w_p(u, v) \geq -\frac{1}{n^2}.$$

We can round w_p to obtain a nonnegative-weight instance and run Dijkstra's algorithm.
After undoing the potential shift, we can recover the exact SSSP distances in G.

Open Problems

- Understand communication complexity of more graph problems.
- Can general matching be solved with $\tilde{O}(n)$ bits of communication?

Thank You!
Q&A