Playing Anonymous Games Using Simple Strategies

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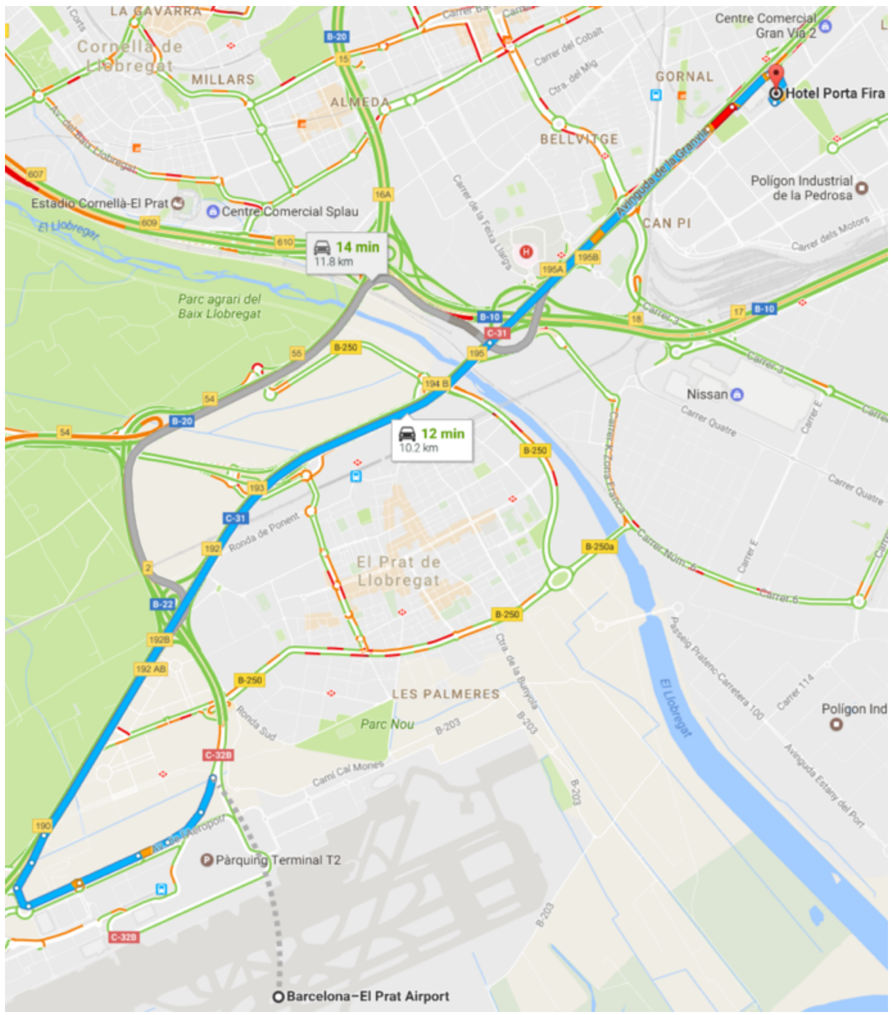
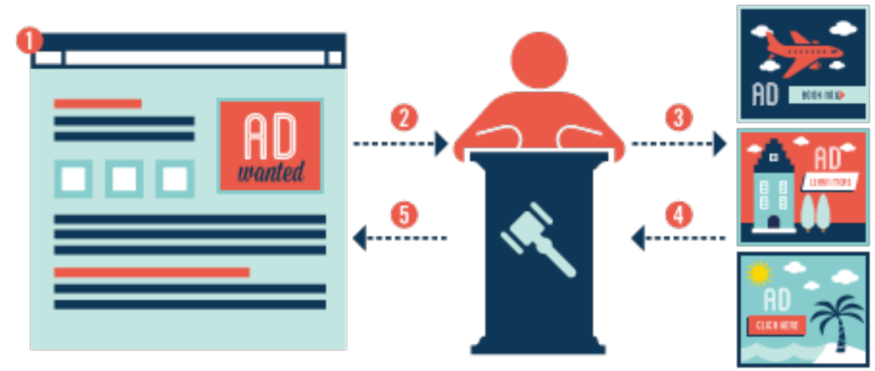
Anonymous Games

- n players, $k = O(1)$ strategies

- Payoff of each player depends on
  - Her identity and strategy
  - The number of other players who play each of the strategy
  - **NOT** the identity of other players

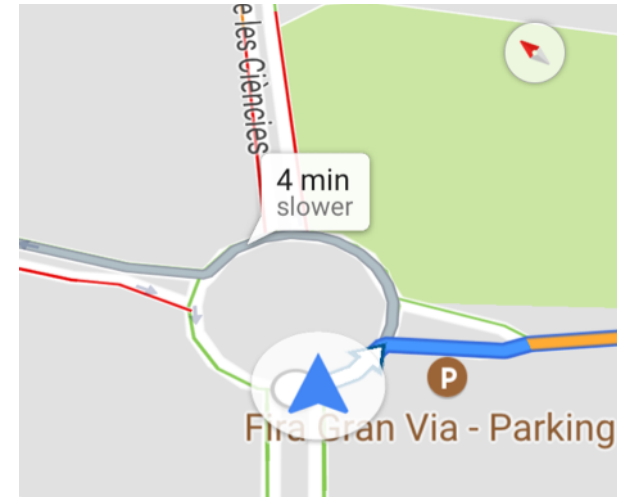
Anonymous Games





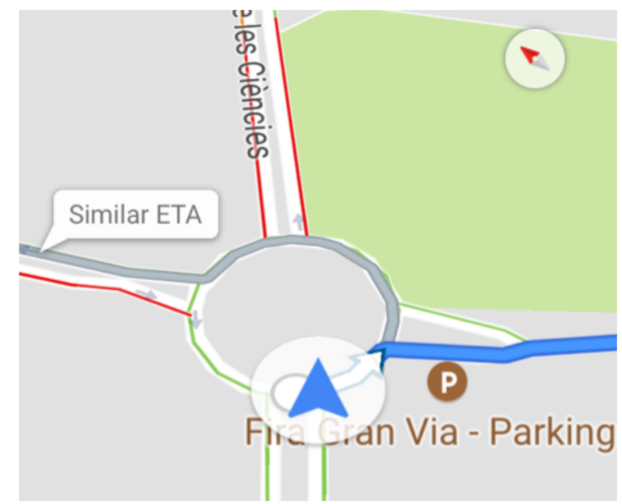
Nash Equilibrium

- Players have no incentive to deviate



- ϵ-Approximate Nash Equilibrium (ϵ-ANE):

  Players can gain at most ϵ by deviation



Previous Work

ϵ-ANE of n-player k-strategy anonymous games:

- [DP'08]: First PTAS $n^{(k/\epsilon)^{O(k^3)}}$
- [CDO'14]: PPAD-Complete when $\epsilon = 2^{-n^c}$ and $k = 5$

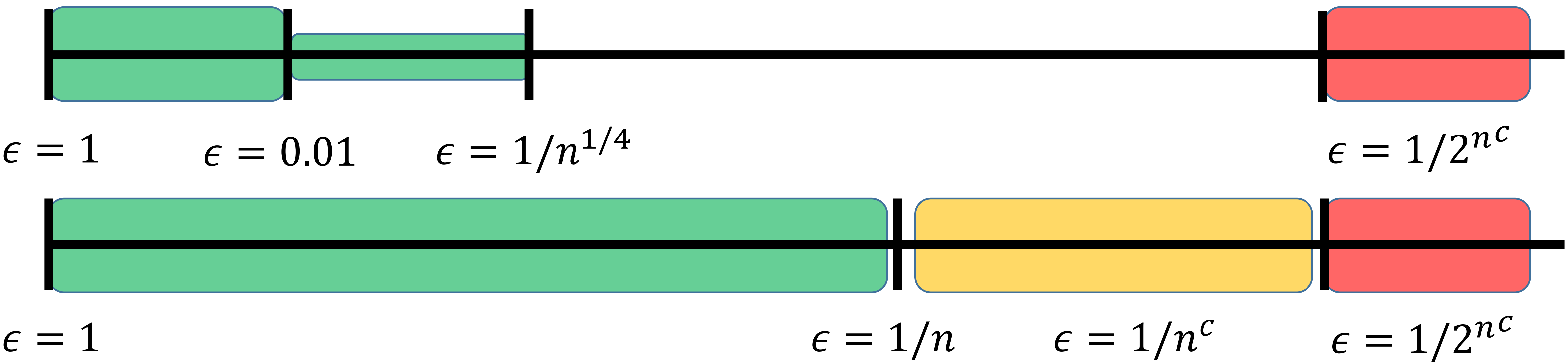
How small can ϵ be
so that an ϵ-ANE
can be computed in polynomial time?

	Running time	ϵ	# of strategies
[DP'08a]	$n^{(k/\epsilon)^{O(k^3)}}$		$k > 2$
[CDO'14]	PPAD Complete	$\epsilon = 2^{-n^c}$	$k = 5$
[DP'08b]	$\mathrm{poly}(n) \cdot (1/\epsilon)^{O(\log^2(1/\epsilon))}$		$k = 2$
[GT'15]	$\mathrm{poly}(n)$	$\epsilon = n^{-1/4}$	$k = 2$
[DKS'16a]	$\mathrm{poly}(n) \cdot (1/\epsilon)^{O(\log(1/\epsilon))}$		$k = 2$
[DKS'16b] [DDKT'16]	$n^{\mathrm{poly}(k)} \cdot (1/\epsilon)^{k \log(1/\epsilon)^{O(k)}}$		$k > 2$

Our Results

Fix any $k > 2, \delta > 0$

- First poly-time algorithm when $\epsilon = \frac{1}{n^{1-\delta}}$
- A poly-time algorithm for $\epsilon = \frac{1}{n^{1+\delta}}$ $\implies$ FPTAS



Our Results

Fix any $k > 2, \delta > 0$

- First poly-time algorithm when $\epsilon = \frac{1}{n^{1-\delta}}$
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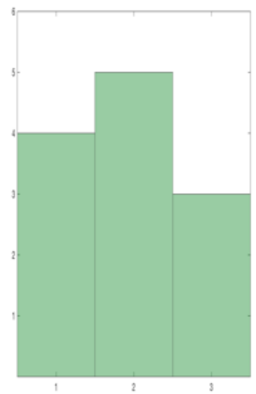
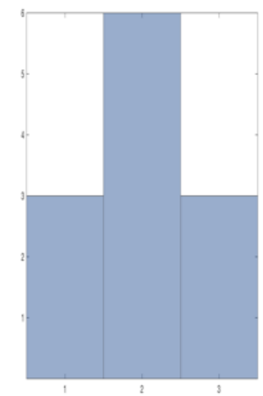
- A faster algorithm that computes an $\epsilon \approx \frac{1}{n^{1/3}}$ equilibrium

Anonymous Games

- Player i's payoff when she plays strategy a

$u_a^i($  $) = 0.3$ $u_a^i($  $) = 0.7$

  - $u_a^i \colon \Pi_{n-1}^k \to [0, 1]$
  - $\Pi_{n-1}^k = \{(x_1, \ldots, x_k) \mid \sum_i x_i = n - 1\}$

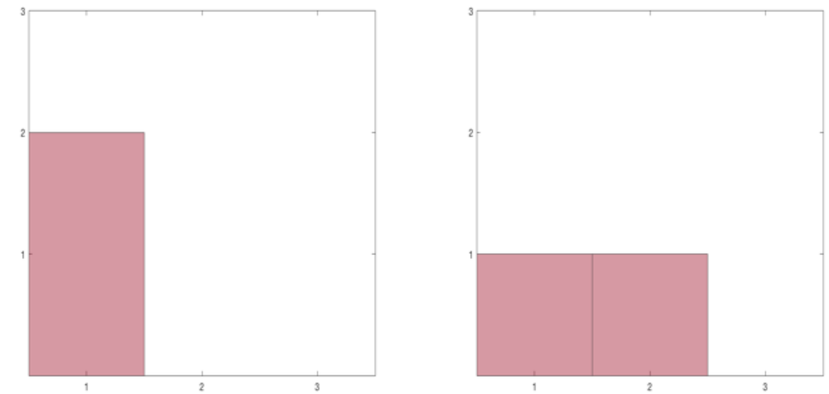
Poisson Multinomial Distributions

- k-Categorical Random Variable (k-CRV) X_i is a vector random variable $\in$ {k-dimensional basis vectors}

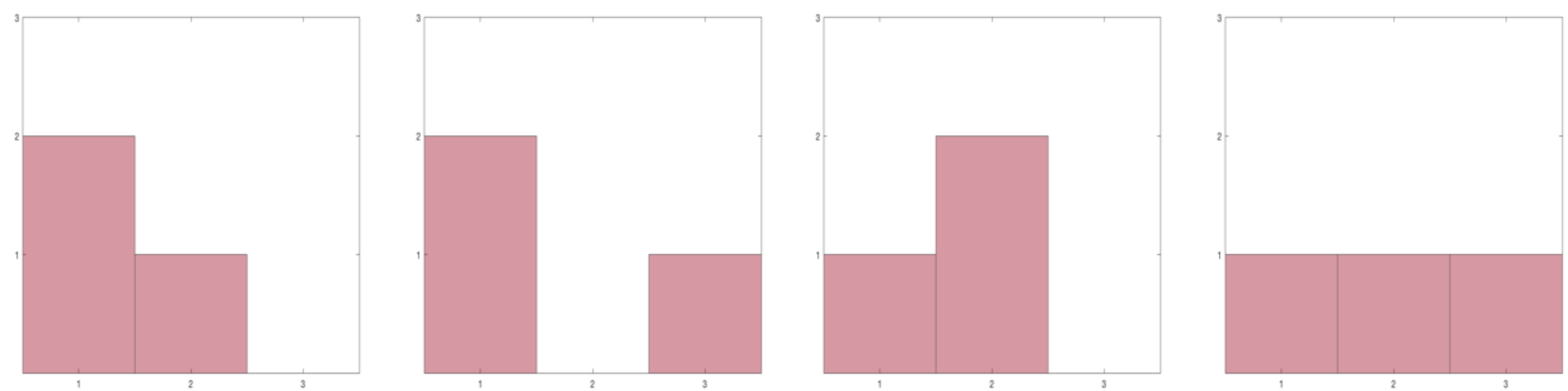
- An (n, k)-Poisson Multinomial Distribution (PMD) is the sum of n independent k-CRVs $X = \sum X_i$

Player 1 plays strategy 1

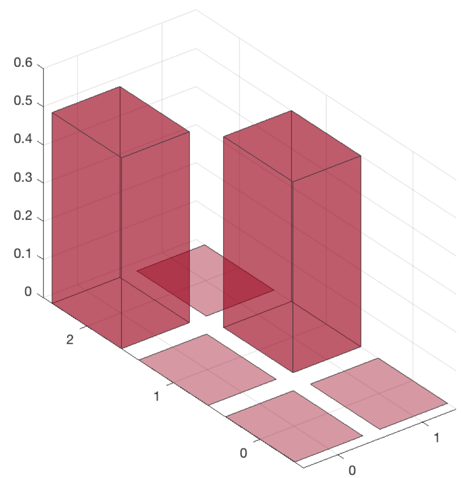
Player 2 plays strategy 1 or 2

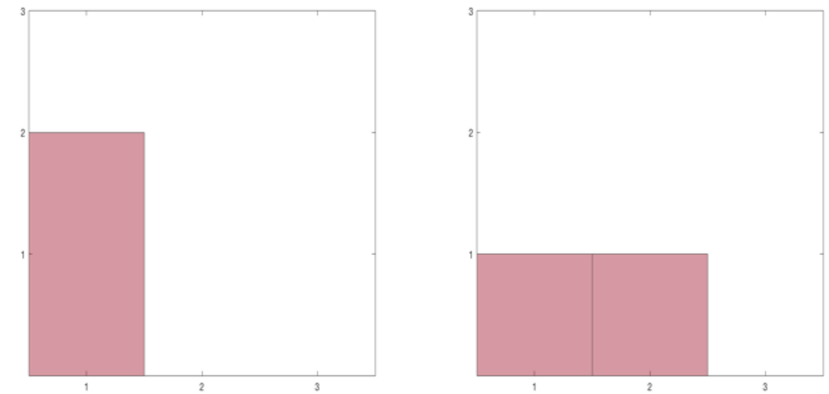


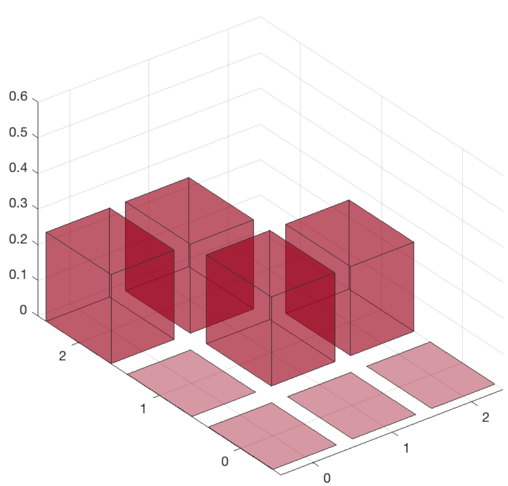
Player 3 plays strategy 2 or 3

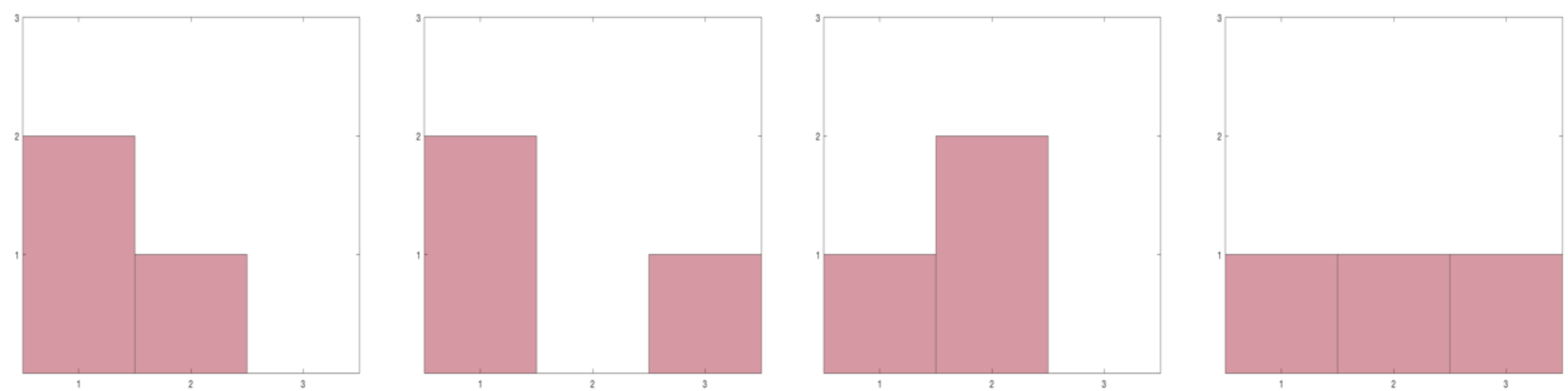


Poisson Multinomial Distributions









Poisson Multinomial Distributions (PMDs)

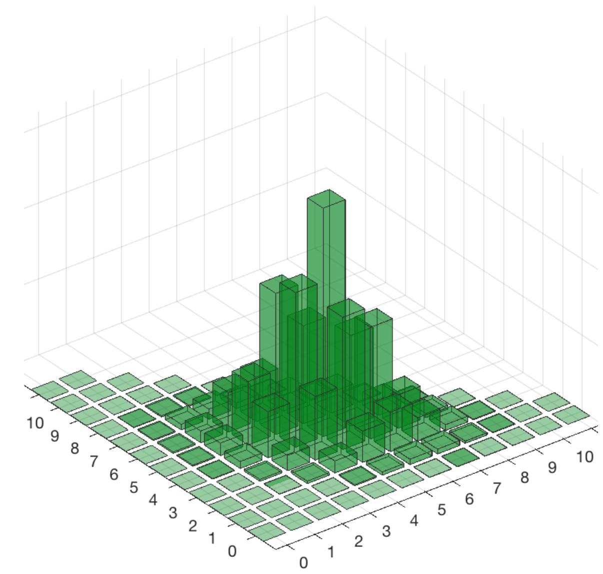
=

Sum of independent random (basis) vectors

=

Mixed strategy profiles of anonymous games

Better understanding of PMDs



Faster algorithms for ϵ-ANE of anonymous games

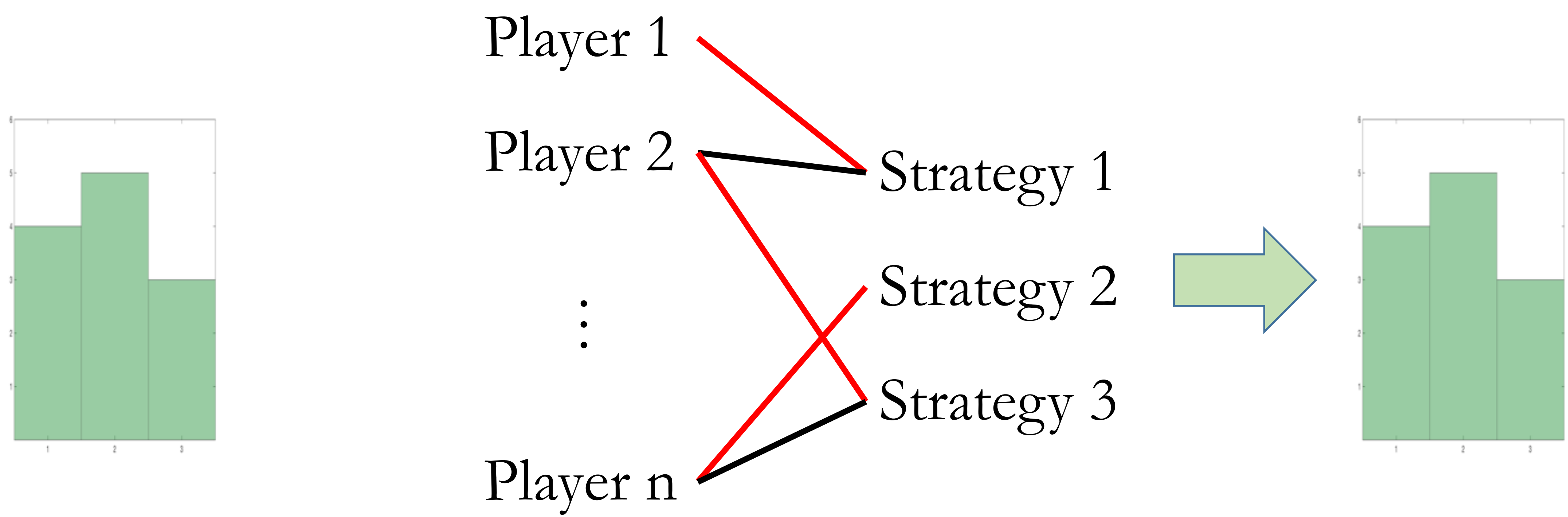
Our Results

Fix any $k > 2, \delta > 0$

- First poly-time algorithm when $\epsilon = \frac{1}{n^{1-\delta}}$
- A poly-time algorithm for $\epsilon = \frac{1}{n^{1+\delta}}$ $\implies$ FPTAS

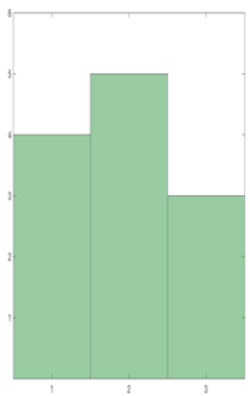
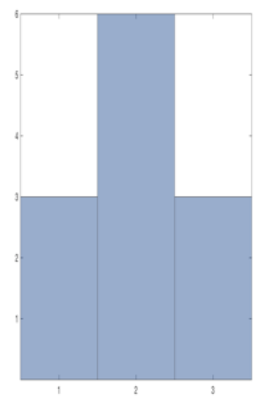
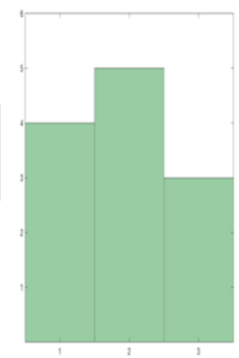
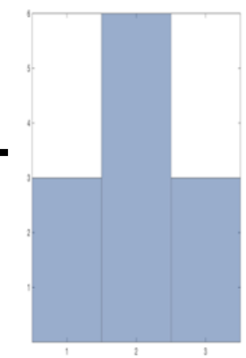
- A faster algorithm that computes an $\epsilon \approx \frac{1}{n^{1/3}}$ equilibrium

Pure Nash Equilibrium



Lipschitz Games

- An anonymous game is λ-Lipschitz if

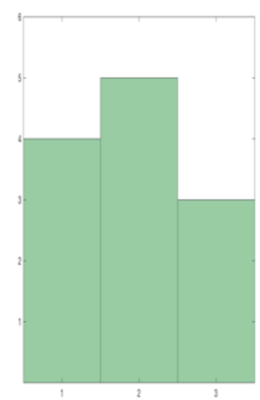
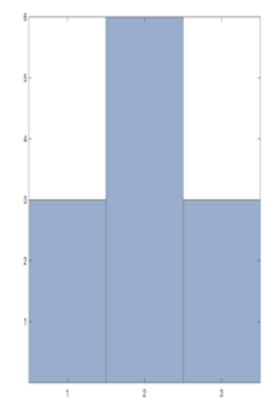
$|u_a^i($  $) - u_a^i($  $)| \leq \lambda \|$  $-$  $\|_1$

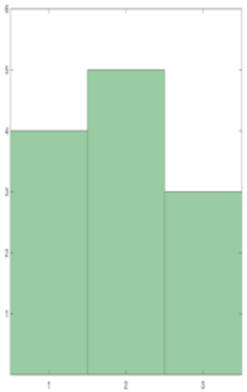
- [DP'15, AS'13] Every λ-Lipschitz k-strategy anonymous game admits a $(2k\lambda)$-approximate pure equilibrium

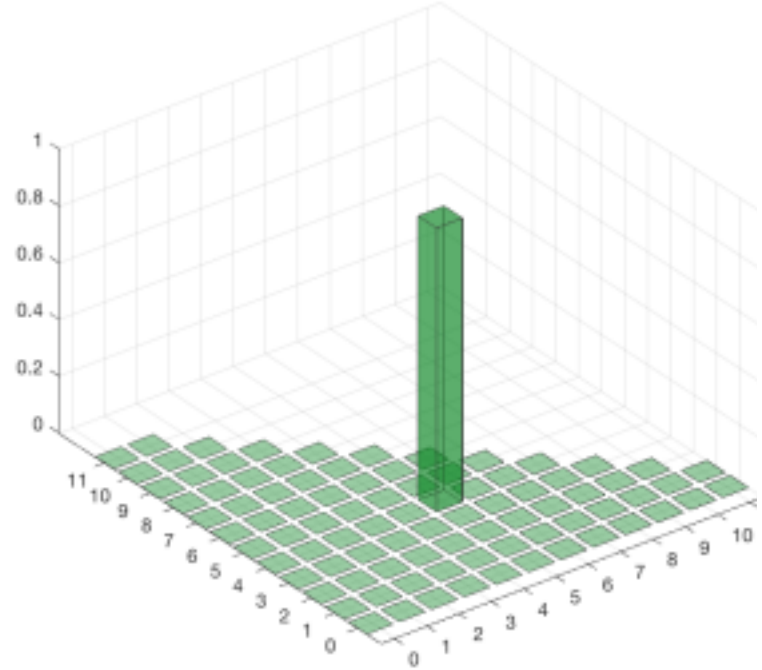
Lipschitz Games

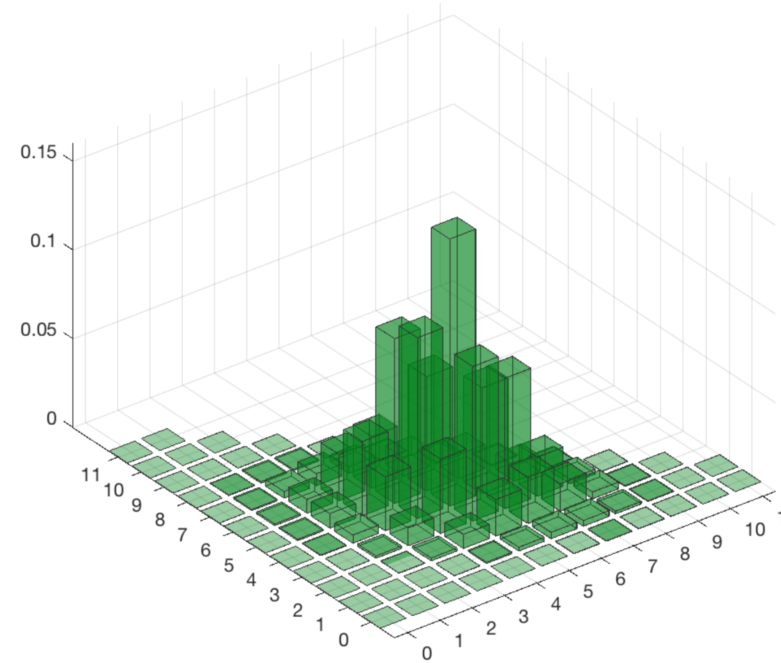
- $(2k\lambda)$-approximate pure equilibrium

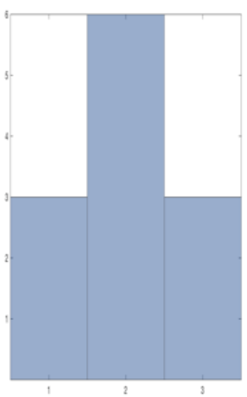
Bad case: $\lambda = 1$

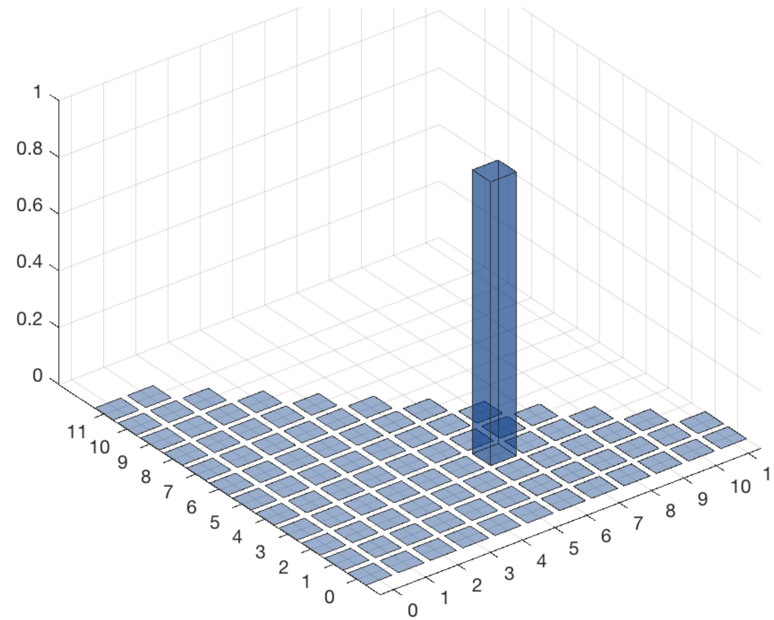
$u_a^i($  $) = 0$ $u_a^i($  $) = 1$

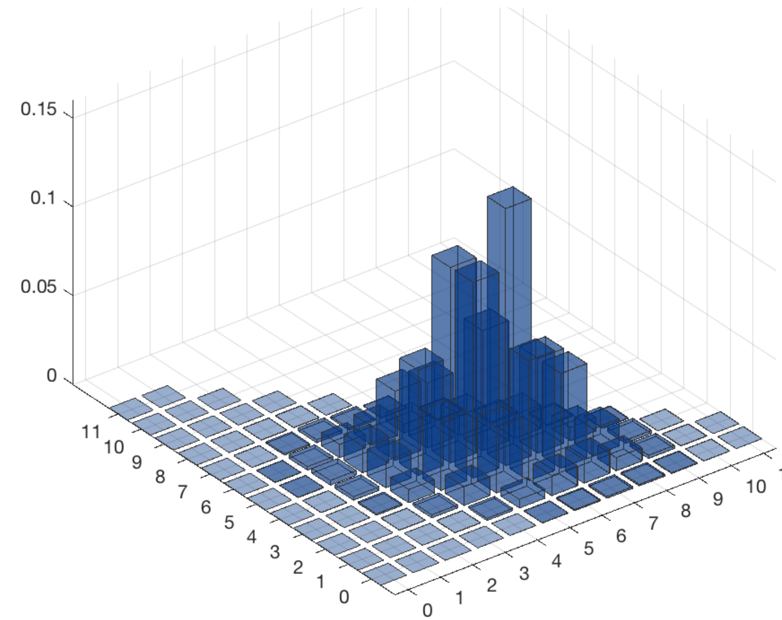
$u_a^i($  $) = 0$



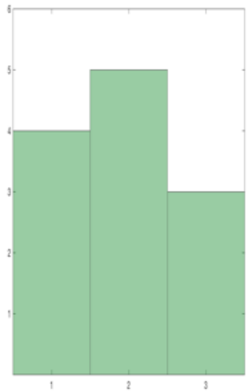


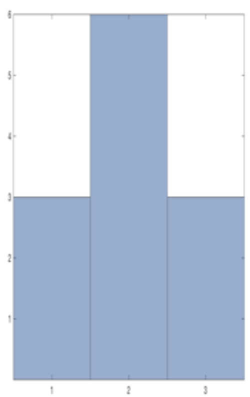
$u_a^i($  $) = 1$

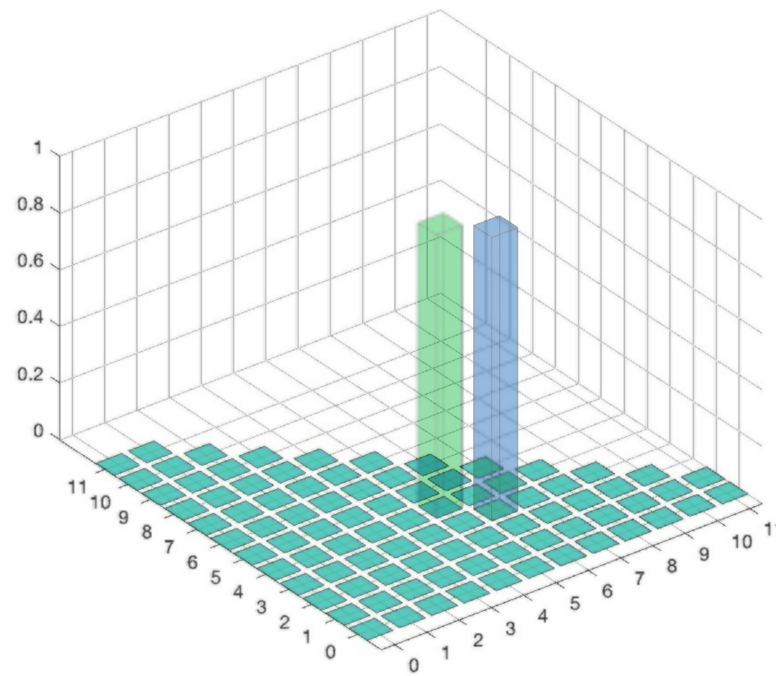




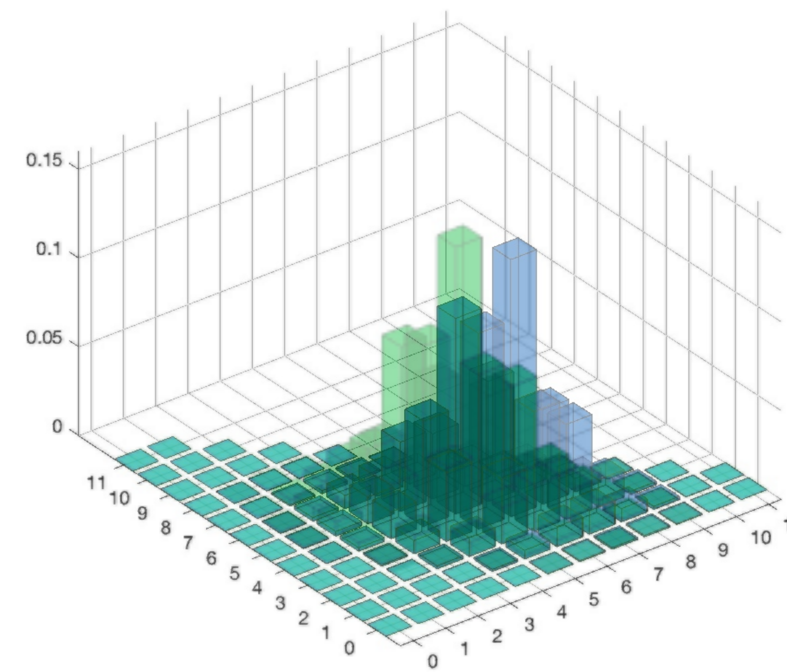
$$d_{TV}(P,Q) := (1/2) \cdot \|P - Q\|_1$$

$u_a^i($  $) = 0$

$u_a^i($  $) = 1$



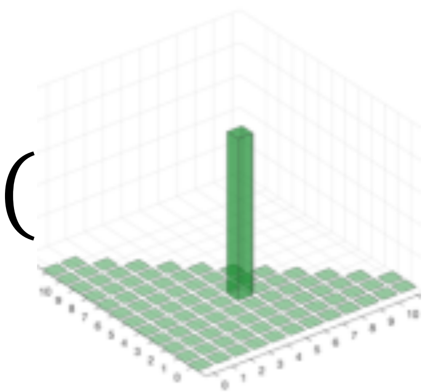
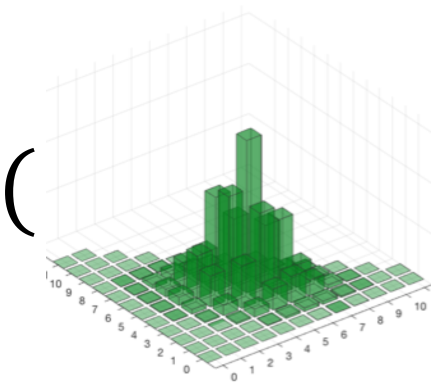
$d_{TV} = 1$



$d_{TV} \ll 1$

Smoothed Game [GT'15]

- Given a game G, construct a new game G_δ

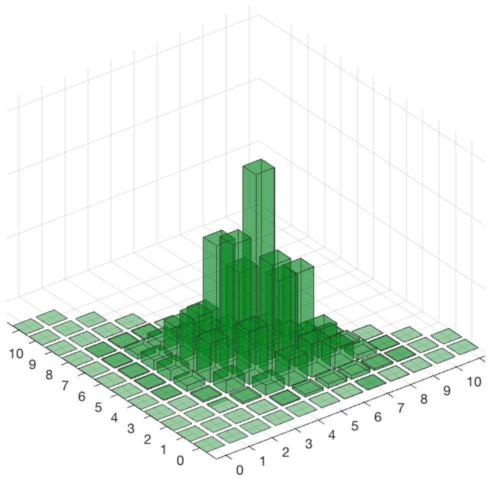
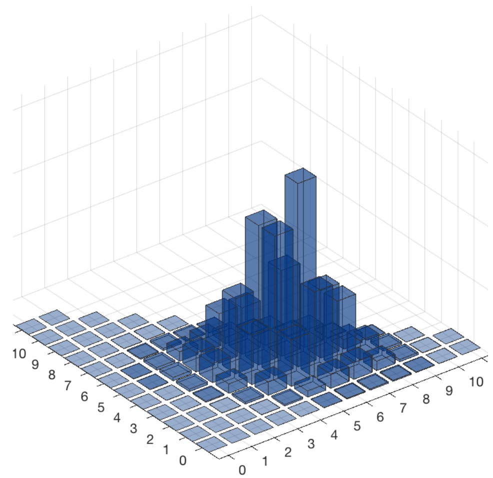
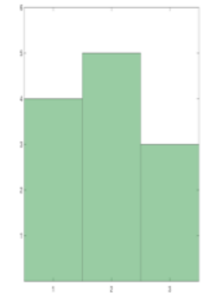
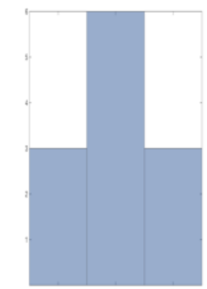
$u'($  $) = \mathbb{E}[u($  $)]$

- G_δ is $\tilde{O}\left(\frac{1}{\sqrt{n\delta}}\right)$-Lipschitz

$\tilde{O}(1/n^{1/3})$-ANE in Polynomial Time

- A $(2k\lambda)$-ANE of G_δ is a $(2k\lambda + \delta)$-equilibrium of G
  - Gain at most $2k\lambda$ by switching to $\left(1 - \delta, \frac{\delta}{k-1}, \dots, \frac{\delta}{k-1}\right)$
  - Gain at most $(2k\lambda + \delta)$ by switching to $(1, 0 \dots 0)$

- $\lambda = \tilde{O}\left(\frac{1}{\sqrt{n\delta}}\right) \implies \epsilon = \tilde{O}\left(\frac{1}{\sqrt{n\delta}}\right) + \delta = \tilde{O}\left(\frac{1}{n^{1/3}}\right)$

$d_{TV}($  ,  $) \leq \tilde{O}(\frac{1}{\sqrt{n\delta}}) \|$  $-$  $\|_1$

$\wr\wr$ $\qquad$ $\wr\wr$

$\mathcal{N}(\mu_1, \Sigma_1) \approx \mathcal{N}(\mu_2, \Sigma_2)$

- Size-free multivariate Central Limit Theorem [DKS'16]:
  an (n, k)-PMD is $\mathbf{poly}(k/\sigma)$ close to discrete Gaussians
- Two Gaussians with similar mean and variance are close

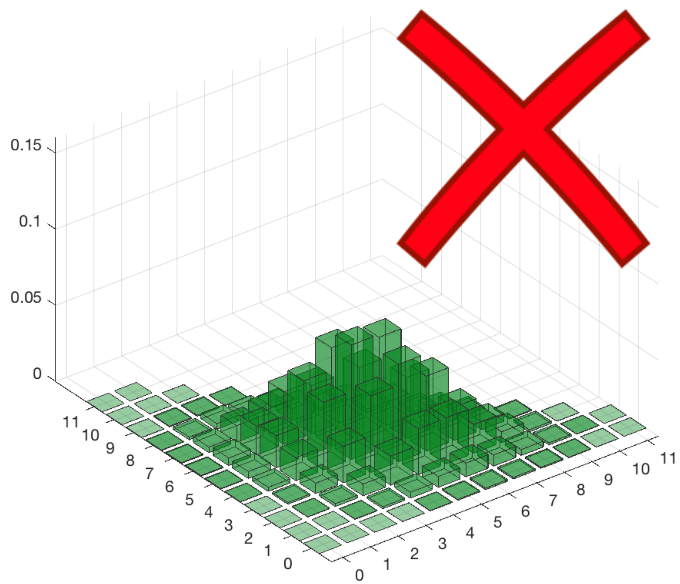
Our Results

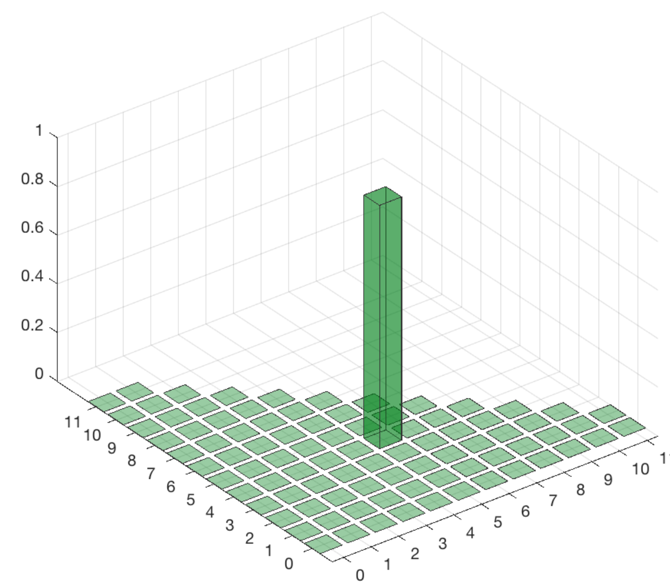
Fix any $k > 2, \delta > 0$

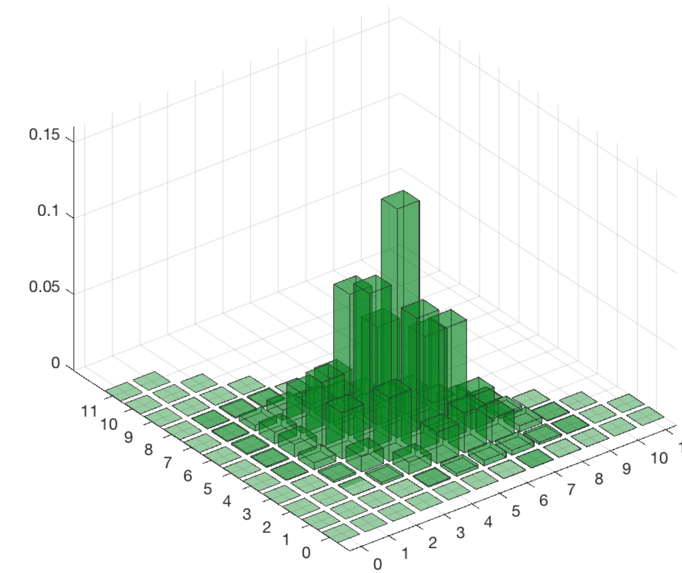
- First poly-time algorithm when $\epsilon = \frac{1}{n^{1-\delta}}$
- A poly-time algorithm for $\epsilon = \frac{1}{n^{1+\delta}}$ $\Longrightarrow$ FPTAS

- A faster algorithm that computes an $\epsilon \approx \frac{1}{n^{1/3}}$ equilibrium

$O(1/n^{0.99})$-ANE


0.15
0.1
0.05
0
11 10 9 8 7 6 5 4 3 2 1 0
0 1 2 3 4 5 6 7 8 9 10 11


1
0.8
0.6
0.4
0.2
0
11 10 9 8 7 6 5 4 3 2 1 0
0 1 2 3 4 5 6 7 8 9 10 11


0.15
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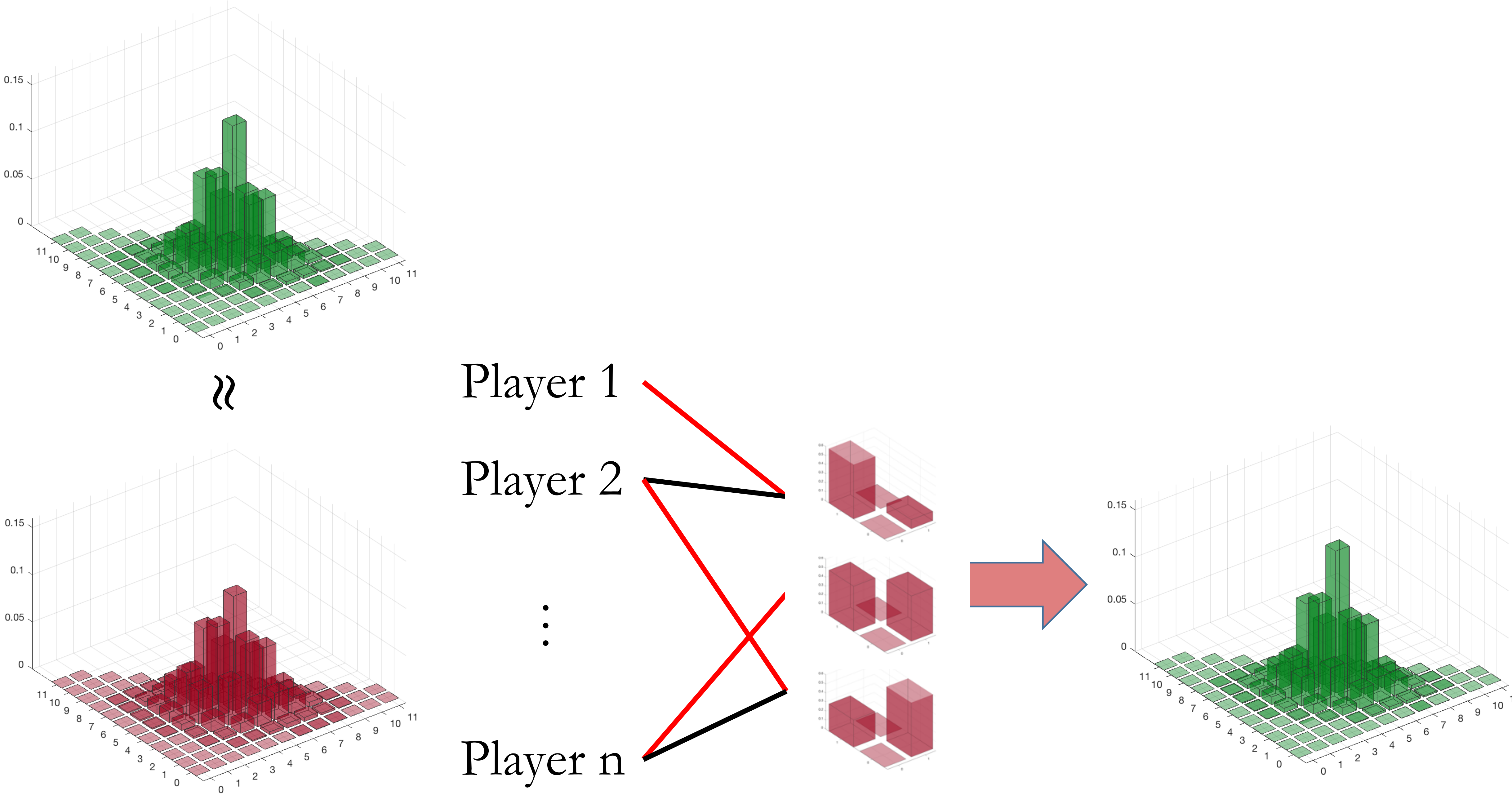
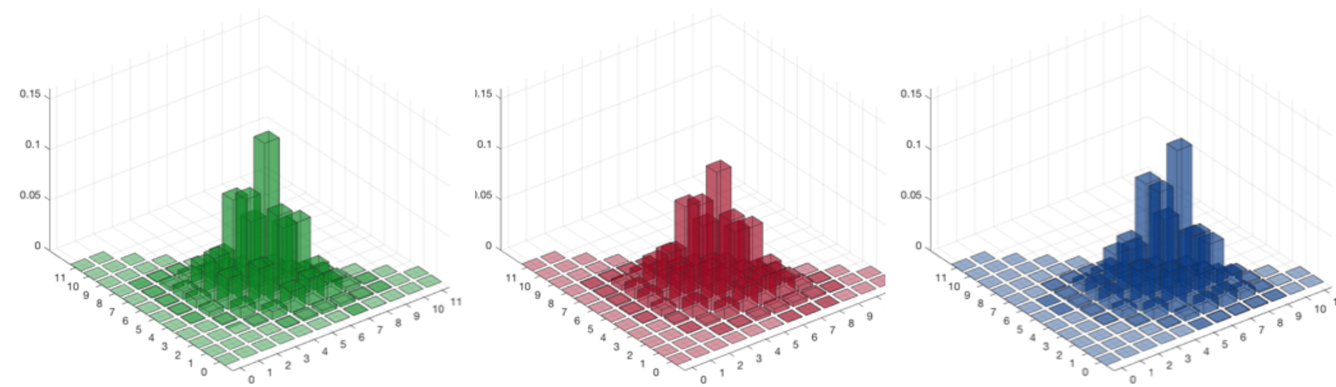
11 10 9 8 7 6 5 4 3 2 1 0
0 1 2 3 4 5 6 7 8 9 10 11

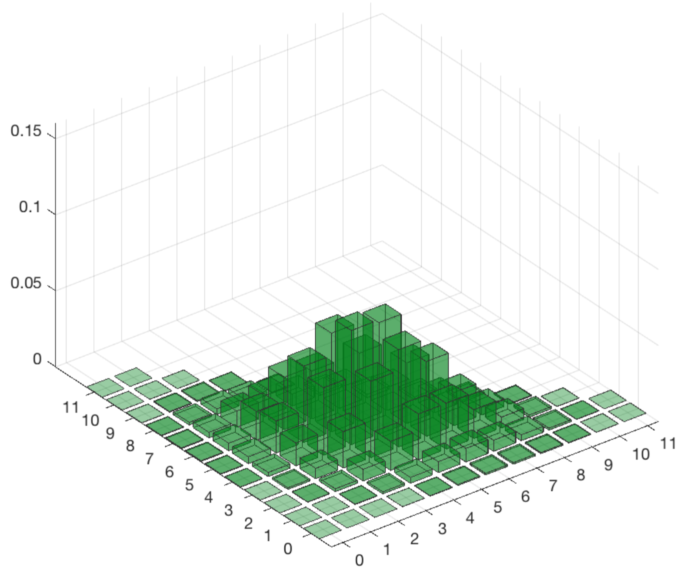

Player 1
Player 2
Player n

Quasi-PTAS when $\epsilon = O(1/n^c)$

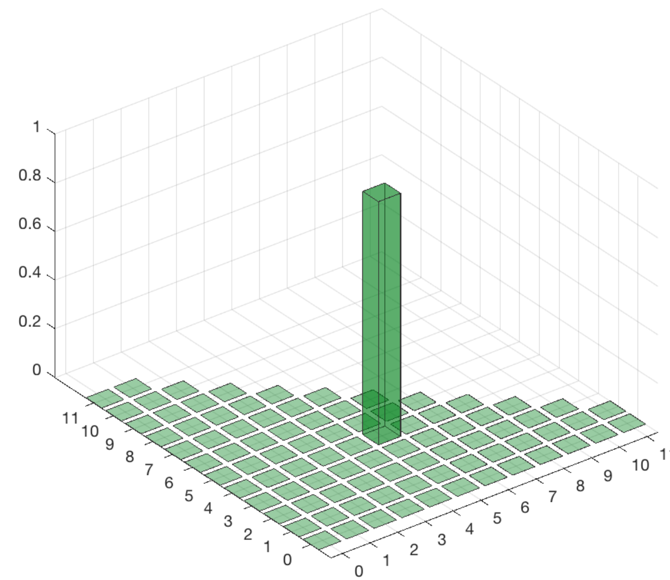
- Small d_{TV} $\Longrightarrow$ Similar payoffs



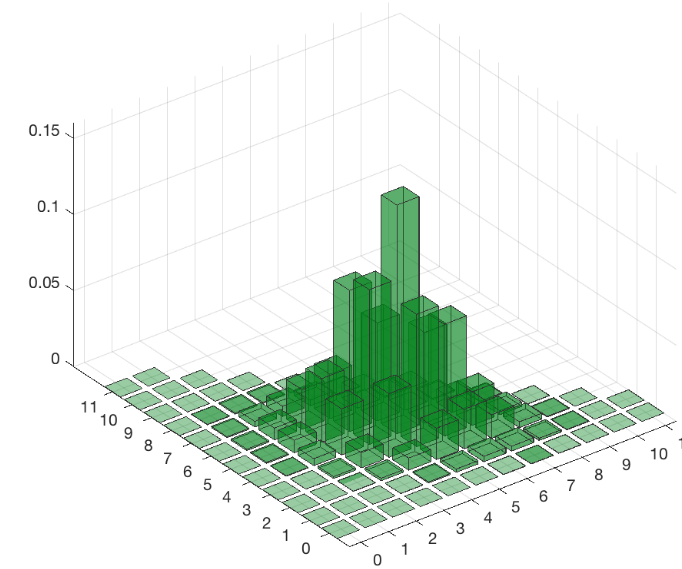
- Limitation:
  - Cover-size lower bound [DKS'16]: even when $k = 2$
    Any proper ϵ-cover S must have $|S| \geq n\ (1/\epsilon)^{\Omega(\log(1/\epsilon))}$



$\epsilon = 1/n^{1/3}$
Two moments



$\log(1/\epsilon)$ moments



$\epsilon = 1/n^{0.99}$
$O(1)$ moments

Moment Matching Lemma

- For two PMDs to be ϵ-close in d_{TV} [DP'08, DKS'16] need first $\log(1/\epsilon)$ moments to match
- We provide quantitative tradeoff between
  - The number of moments we need to match
  - The size of the variance

Moment Matching Lemma

- Multidimensional Fourier transform
  - Exploit the sparsity of the Fourier transform

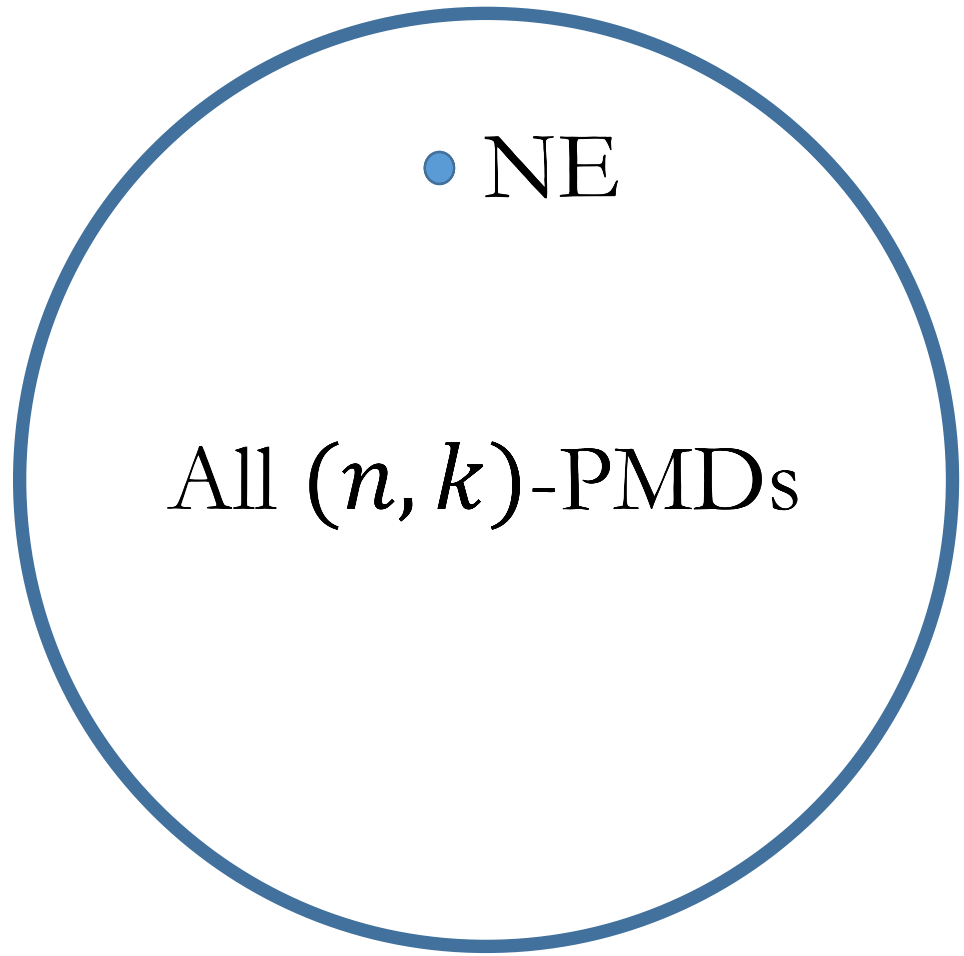
- Taylor approximations of the log Fourier transform
  - Large variance $\Longrightarrow$ Truncate with fewer terms

$O(1/n^{0.99})$-ANE in Polynomial Time

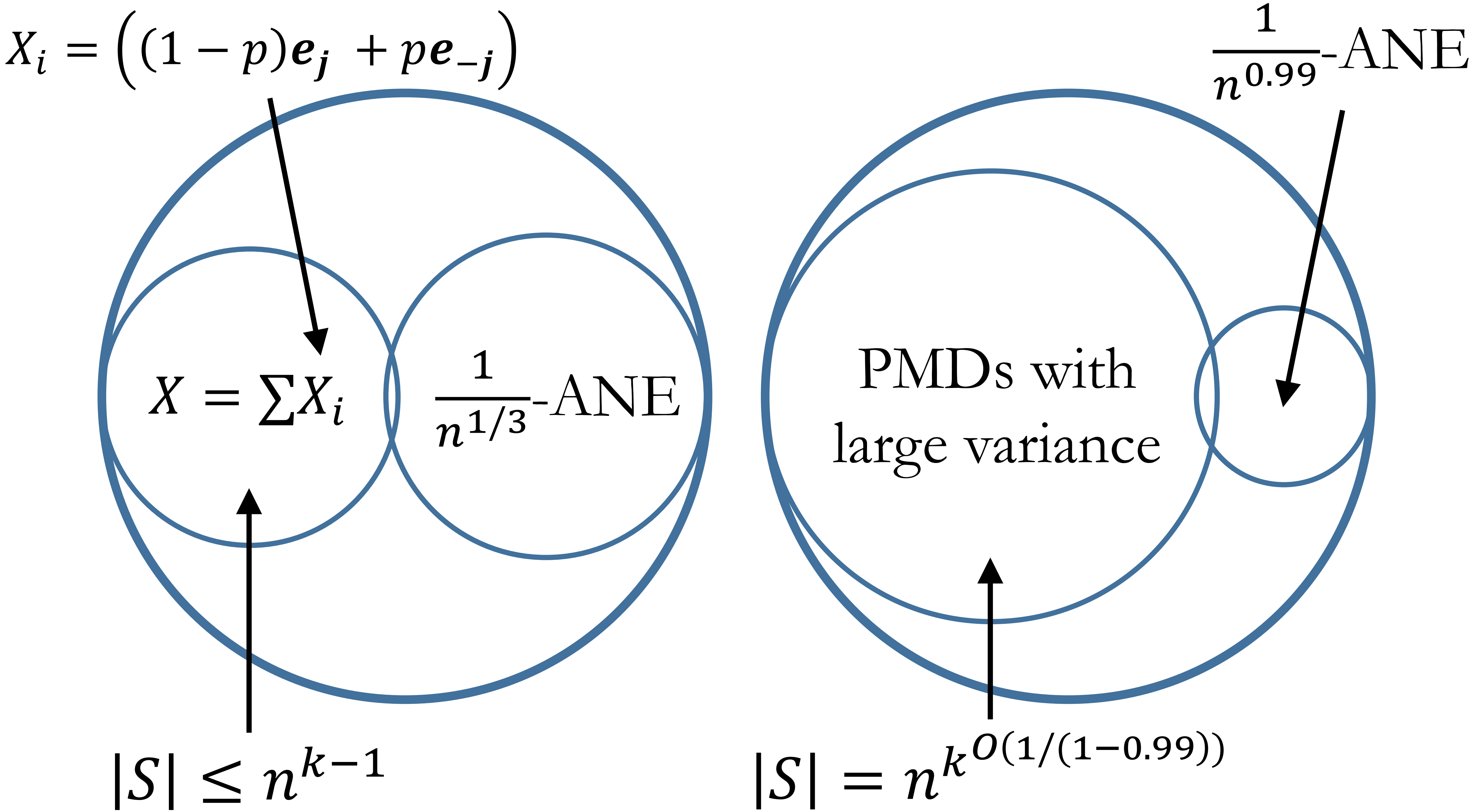
- There always exists an equilibrium with variance

$$\epsilon n = n^{-0.99} \cdot n \;=\; n^{0.01}$$

- Construct a poly-size ϵ-cover of large variance PMDs
    - Polynomial-size: Match only degree $O(1)$ moments



$$|S| \geq n\,(1/\epsilon)^{\Omega(\log(1/\epsilon))}$$

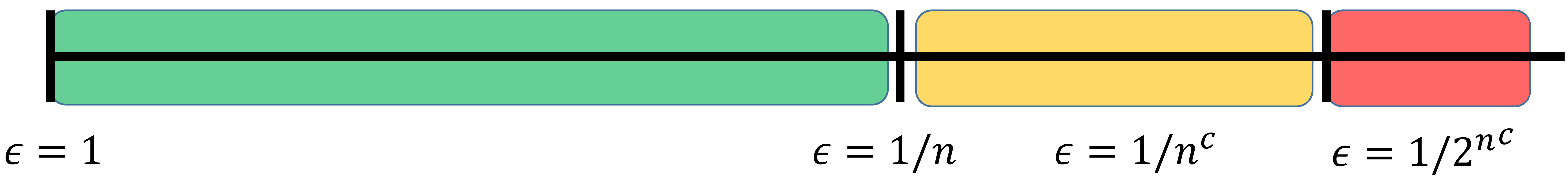


$X_i = \left((1-p)\boldsymbol{e}_j + p\boldsymbol{e}_{-j}\right)$
$X = \sum X_i$
$\frac{1}{n^{1/3}}$-ANE
$|S| \leq n^{k-1}$
$\frac{1}{n^{0.99}}$-ANE
PMDs with large variance
$|S| = n^{k^{O(1/(1-0.99))}}$

Conclusion

Computing ϵ-ANE of n-player anonymous games

- First poly-time algorithm when $\epsilon = \frac{1}{n^{1-\delta}}$
  - New moment-matching lemma for PMDs
- A poly-time algorithm for $\epsilon = \frac{1}{n^{1+\delta}}$ $\implies$ FPTAS



Open Problems