CSCI 0500: Data Structures, Algorithms, and Intractability (Fall 2025)

Assignment 7

Due at 11:59pm ET, Friday, Dec 19

1. (1 point) In this question, we will prove hardness results for the problems of minimum vertex cover, maximum independent set, and maximum clique.

   Let $G = (V, E)$ be an undirected graph. A vertex cover of G is a set of nodes that includes at least one endpoint of every edge. An independent set of G is a set of nodes such that no two nodes in the set are connected by an edge. A clique of G is a set of nodes such that every pair of nodes in the set is connected by an edge. Consider the decision versions of these problems:

   Definition 1 (Vertex Cover (VC))**.** Given as input a graph $G = (V, E)$ and an integer k, does G have a vertex cover of size at most k?

   Definition 2 (Independent Set (IS))**.** Given as input a graph $G = (V, E)$ and an integer k, does G have an independent set of size at least k?

   Definition 3 (Clique)**.** Given as input a graph $G = (V, E)$ and an integer k, does G have a clique of size at least k?

   Lemma 1. *We have* 3-SAT $\leq_P$ IS*, and consequently, Independent Set is NP-hard.* [1]

   You are encouraged to try to prove this lemma, but it is not required for this assignment.

   Your task is to reduce Independent Set to the other two problems. For Parts (a) and (b), you do not need to prove the correctness of your reductions.

   (a) Show a Karp reduction from Independent Set to Vertex Cover.

   (b) Show a Karp reduction from Independent Set to Clique.

   All three problems are in NP, because "yes" solutions can be verified in polynomial time. Consequently, all three problems are NP-complete.

2. (1 point) Decide whether each of the following statements is true or false. Justify your answer.

   (a) If $X \leq_P Y$ and $Y \in$ P, then $X \in$ P.

   (b) If an NP-hard problem X has a polynomial-time algorithm, then P = NP.

   (c) If $X \in$ NP-complete and $Y \in$ NP-hard, then $X \leq_P Y$.

   (d) If 3-SAT $\leq_P X$, then X is NP-complete.

[1] We write $A \leq_P B$ if there is a Karp reduction from problem A to problem B.

3. (1 point) In this question, we will explore the halting problem.

   There is no required task for this question. You will receive full credit on this question if you make a submission.