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Dynamics of Physical Systems

For our purposes, a *process model* is a device that, given certain information about the state of a physical system, enables us to determine certain other information about that system. The device usually includes some mathematical characterization of the system's properties and how they relate to one another. It also includes some sort of a calculus whereby an engineer or a machine can compute the predictions of the model given some initial conditions.

Process models are used by engineers to design control systems. In some cases, the process model is used only to evaluate a given controller. In other cases, the process model becomes an integral part of the control system. In this chapter, we consider a few of the large number of process modeling techniques available to the engineer and develop some notation for describing process models that will be used in subsequent chapters.

2.1 Process Models

To construct a model for a process, we have to identify those properties of the world that determine the behavior of the process. First, there are those properties that prompted our interest in the process to begin with. In the case of the tank-filling process described in Chapter 1, we are primarily interested in the height of the fluid in the tank. Second, there are those properties that affect the properties that we are interested in. In order to account for the level of fluid in the tank, we have to know the dimensions of the tank, the flow characteristics of the input and output pipes, and the position of the valves. It is easy to underestimate the difficulty of this part of the modeling task.

Textbooks typically just give the student the set of physical properties that he or she needs to be concerned with. There is an implicit assumption that these are all and only the properties that need to be considered. How do we know that the temperature of the fluid does not affect the height of the fluid in the tank? Well, of course, we don't know this. The temperature may affect the fluid height by changing the rate at which the fluid evaporates; however, given that the temperature does not vary substantially, the effect of temperature on fluid height is negligible.

Almost any property of the world *can* have an impact on the level of the fluid in the tank; agricultural trends affect global weather patterns that affect local temperature and humidity that ultimately affect fluid height. The predictions made by a particular model are likely to be accurate only if certain assumptions hold. Whether or not to account for a given property of the world in a particular model depends on a number of factors: the magnitude of the effect (*i.e.,* does it result in substantial changes in the properties of interest), the probability of the effect (*i.e.,* do the changes occur with high frequency), and the complexity of the model (*i.e.,* what additional computations are required to account for the property in the model).

This last is particularly important, and, yet, it is often overlooked in evaluating a model. There is often some utility in getting an answer to a question quickly. If this were not the case, you would always want the model that makes the most accurate predictions possible. Given that time has to be taken into account, there is a tradeoff to be made regarding the accuracy of the model and the time that it takes to compute its predictions.

The following sections describe some basic methods for modeling physical processes in control theory. Section 2.2 considers the use of the differential and integral calculi for modeling processes and analyzing the behavior of control systems, focusing on ideas from classical control theory. Section 2.3 considers the general problem of modeling dynamical systems and introduces ideas from linear system theory, drawing upon results from modern control theory.

2.2 Classical Design and Analysis

Much of control theory depends on the use of mathematical models based on the techniques of the integral and differential calculi. These techniques enable the control theorist to model a wide variety of mechanical, electrical, fluid, and thermodynamic systems. By modeling both the controlling process and the process being controlled as a set of differential equations, the control theorist is able to analyze behavior of the combined system and predict the performance characteristics of the controlling process (*e.g.,* how fast the system responds to a disturbance or change in input). In this section, we summarize some of the issues involved in modeling physical systems using the techniques of control theory.

Anyone who has taken a course in differential equations or advanced calculus has seen numerous examples of mathematical models of physical systems. Most introductory texts on the differential calculus include idealized models of population growth, the decay of radioactive materials, and the fluctuation in prices as a function of supply and demand. If you took a physics course, you were early on exposed to Newton's laws of motion. Newton's second law of motion states that the product of a body's mass and the acceleration of its center of mass is proportional to the force acting on the body. Let x be a function that depends on t and denotes the position of the center of mass of the object as measured from some fixed point along a vertical line. Let M be the mass of the object, and $\mathcal{F}$ be the force acting on the object in the direction of travel. The following differential equation

$$M\frac{d^2x}{dt^2} = \mathcal{F} \tag{2.1}$$

is called the *equation of motion* of the body.[1] If we know something about the forces acting on the body, then we can use this equation to make predictions about the motion of the body.

If x is the directed distance upward of the object as measured from the surface of the earth, and v_0 is the object's initial velocity, then, assuming that the only force acting on the object is gravity, Equation 2.1 becomes

$$M\frac{d^2x}{dt^2} = -Mg \tag{2.2}$$

where g is the acceleration due to gravity near the surface of the earth. We can solve this simple second-order differential equation by integrating twice and using the initial conditions to determine the constants of integration. The following formula

$$x(t) = -\frac{1}{2}gt^2 + v_0 t \tag{2.3}$$

describes the position of the object at $t \geq 0$ given the initial conditions

$$x(0) = 0, \qquad \frac{d\,x(0)}{dt} = v_0,$$

[1]To simplify the discussion, we implicitly adopt the standard system of units for measuring mass, distance, and time so that the constant of proportionality is one.

and assuming that the object is propelled upward at time $t = 0$. From Equation 2.3, we can predict the maximum height $(v_0^2/2g)$ reached by the object and the time it takes the object to fall back to the surface of the earth $(2v_0/g)$. Equation 2.3, together with tools of the differential calculus, provides us with a simple model of an object falling through a gravitational field.

We know that Equation 2.3 is only approximate in that it neglects several important influences on objects falling through a relatively dense atmosphere under the influence of gravity. For instance, Equation 2.3 treats gravity as a constant acceleration whereas we know that Newton's inverse square law provides a more accurate estimate of the force due to gravity acting on an object. If the earth is assumed to be a sphere of radius R, and r denotes the distance from the center of mass of the object to the center of the sphere, then

$$M\frac{d^2x}{dt^2} = -\frac{MgR^2}{r^2}$$

can provide a more accurate estimate of the position of the object than that provided by Equation 2.1, especially in the case of an object that travels a significant fraction of the distance R.

We can also account for the damping force exerted on the object by the atmosphere as the object moves along its trajectory. If the damping force is proportional to the object's velocity, and C is the damping constant, then

$$M\frac{d^2x}{dt^2} = -\frac{MgR^2}{r^2} - C\frac{dx}{dt} \tag{2.4}$$

will, at least potentially, provide a better estimate than equations that neglect friction. Potentially, because, having identified that some property of the environment influences a particular process, you still have to determine the form and the magnitude of that influence. There are situations in which the damping force is more nearly proportional to the square or the cube of the velocity. In addition, the damping "constant" may not be constant at all, dependent as it is on the shape of the object and the density of the air through which the object is moving. If you are not careful, you can actually reduce the predictive accuracy of a model by trying to account for additional properties.

As another example of physical modeling, Figure 2.1 shows a block of mass M suspended from the ceiling by a spring and connected by a rigid rod at its base to a damping device called a *dashpot*. The spring counteracts the force of gravity and the dashpot tends to inhibit vertical motion in

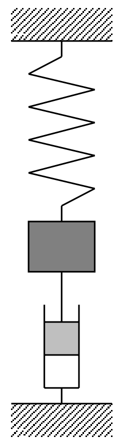

Figure 2.1: A spring-mass-dashpot system

either direction. Suppose that the force exerted by the spring is equal to the product of the distance that the spring is stretched or compressed and K, the spring constant. Let d be the distance past the spring's resting length such that the force of the spring completely offsets the force of gravity, and the block will remain at rest (*i.e.*, $Mg = Kd$). The equation of motion for the block, neglecting the dashpot, is

$$M\frac{d^2x}{dt^2} = Mg - K(x+d) = Kx. \tag{2.5}$$

To account for the dashpot, we assume that the damping action of the dashpot is proportional to the velocity of the block and introduce another term into Equation 2.5. The result is

$$M\frac{d^2x}{dt^2} + C\frac{dx}{dt} + Kx = 0 \tag{2.6}$$

where C is the damping constant.

There are three different solutions to Equation 2.6 depending on whether the quantity C^2 is less than, greater than, or equal to the quantity $4MK$. These solutions correspond to the underdamped, overdamped, or critically damped cases. If $C^2 < 4KM$, then the specific solution to Equation 2.6 that satisfies the initial conditions,

$$x(0) = x_0, \qquad \frac{dx(0)}{dt} = 0,$$

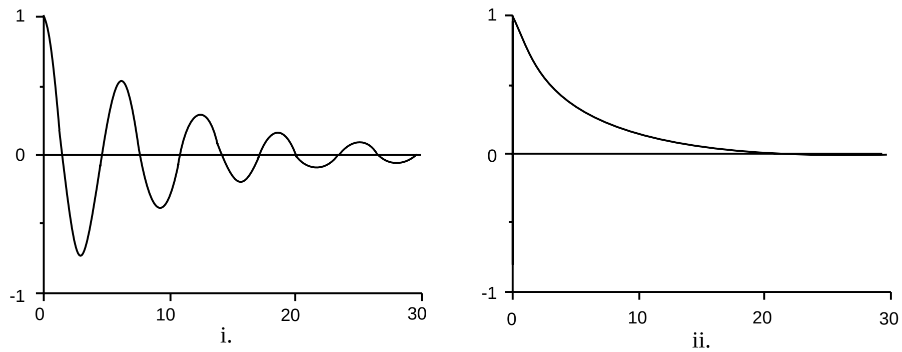

Figure 2.2: Response of the spring-mass-dashpot system in the (i) underdamped and (ii) overdamped cases.

is given by

$$x(t) = x_0 e^{\alpha t}\left(\cos \omega t + \frac{\alpha}{\omega}\sin \omega t\right),$$

where

$$\alpha = \frac{C}{2M}, \qquad \omega = \frac{1}{2M}(4MK - C^2)^{1/2}.$$

In this (the underdamped) case, the mass oscillates about the equilibrium point, its amplitude decreasing exponentially with time as shown in Figure 2.2.i. If $C^2 > 4KM$, then the specific solution to Equation 2.6 satisfying the same initial conditions is given by

$$x(t) = \frac{x_0}{\beta - \alpha}\left(\beta e^{-\alpha t} - \alpha e^{-\beta t}\right),$$

where

$$\alpha = -\frac{1}{2M}\left[-C + (C^2 - 4MK)^{1/2}\right], \quad \beta = -\frac{1}{2M}\left[-C - (C^2 - 4MK)^{1/2}\right].$$

Figure 2.2.ii illustrates the behavior of the resulting overdamped system. The important thing to note here is that, assuming M is fixed, we can vary K and C to achieve different behaviors.

Control theorists are often interested in how a physical system responds to a particular input signal. The *step input*, corresponding to a fixed-size instantaneous change in the reference or a disturbance, provides a convenient basis for comparing performance. In the case of the spring-mass-dashpot, a step input might correspond to the block being displaced from its equilibrium point or given some initial velocity. Equation 2.6 might serve as a simple model for an automobile shock absorber. The input signal would

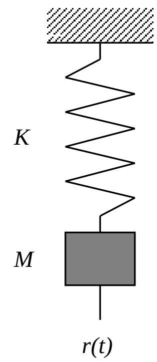

Figure 2.3: An external force acting on a spring-mass system

correspond to a force acting on the mass (*e.g.,* the automobile hitting a bump in the road). The engineer designing such a system is interested in the characteristics of the output signal corresponding to the changes in the position of the mass. In particular, the engineer wants to know whether or not the control system he or she designs is a *stable system*. A system is said to be stable if its response to a bounded input is itself bounded. In the case of our spring-mass-dashpot system, if we displace the mass a small amount from its equilibrium point, it will eventually return to that point. Similarly, if we give the mass some small initial velocity, it will also eventually return to its equilibrium point.

Unstable systems can manifest undesirable and sometimes violent behavior (*e.g.,* thermal runaway in a nuclear power plant). Suppose that we eliminate the dashpot from our spring-mass-dashpot system and introduce an additional, external force acting on the mass as pictured in Figure 2.3. Suppose that the external force is periodic of the form,

$$r(t) = R \sin \omega t,$$

where R is a positive constant. The equation of motion is

$$M \frac{d^2 x}{dt^2} + Kx = R \sin \omega t.$$

If $\omega = (K/M)^{1/2}$, then the amplitude of the oscillations will increase due to the phenomenon of resonance [12]. The model predicts that the oscillations will increase indefinitely, but, of course, there will come a point past which

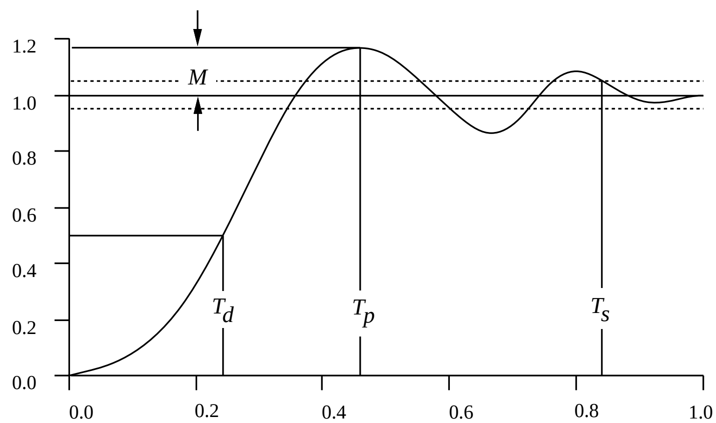

Figure 2.4: Transient response to a step input indicating T_d (*delay time*) the time required for the controlled variable to reach 50% of the target, T_s (*settling time*) the time required for the controlled variable to achieve and maintain a value $\pm 5\%$ of the target, T_p (*peak time*) the time at which the controlled variable achieves the largest value above the target, and M (*peak overshoot*) the largest value of the controlled variable above the target.

the mathematical model is no longer appropriate and other physical properties will come into play (*e.g.,* the spring breaks or the device generating $r(t)$ reaches saturation).

The behavior of a system in transition from one stable state to another as a result of a step input is referred to as the system's *transient response* to a step input. Transient response characteristics include the system's *settling time* (*i.e.,* the amount of time it takes the system to achieve a state in which the value of the controlled variable is within some small percentage of the target value), the system's *steady-state error* (*i.e.,* the percent error of the system in the limit), and the system's *overshoot* (*i.e.,* the maximum past the target that the system achieves in responding to step input). Figure 2.4 illustrates some of the important characteristics of a system's transient response to step input [7, 15].

Peak overshoot is a particularly important transient response characteristic in a number of applications. In some cases, the sort of underdamped behavior shown in Figure 2.2.i is unacceptable. In attempting to restore equilibrium, the system overshoots the target or equilibrium point. In the case of a robot arm positioning a part, overshoot might correspond to the part striking a surface. In the case of the liquid-level system of Chapter 1, overshoot might mean that the level of fluid in the tank goes above the top of the tank, spilling fluid on the floor.

A good deal of control theory is concerned with analyzing the performance of control systems with regard to criteria such as stability, settling time, steady-state error, and overshoot. One way to analyze a control system is to build a mathematical model as a system of differential equations, solve the equations, and then examine the behavior of the system in the time domain. This is essentially what was done in our analysis of the spring-mass-dashpot system above. This method of analysis can be complicated by the fact that the equations for any reasonably complex control system are likely to be difficult to solve, and, in order to find parameters for the control system that provide good performance, it may be necessary to look at a large number of special cases. While there exist effective methods for analyzing control systems in the time domain, one of the great successes of what is called *classical control theory* has been the development of mathematical techniques that enable an engineer to recast a control problem as a problem in the frequency domain. Most of these techniques rely on the use of the *Laplace transform*.

The Laplace transform enables the control theorist to avoid working with differential equations by replacing these generally difficult-to-solve equations with simpler algebraic equations. Since the Laplace transform exists for many linear differential equations encountered in control systems design, methods based upon the use of the Laplace transform are widely employed in the analysis of control systems. The Laplace transform of a function of time, $f(t)$, is defined as

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \mathcal{L}(f(t)). \tag{2.7}$$

The Laplace transform of the derivative of a function can be obtained from Equation 2.7 using integration by parts

$$\mathcal{L}\left(\frac{df(t)}{dt}\right) = s\mathcal{L}(f(t)) - f(0).$$

However, it is usually not necessary to derive the Laplace transform of a function every time that the engineer is faced with a new problem. Tables of functions and their Laplace transforms have been compiled for most functions commonly encountered in engineering applications.

The Laplace transform of a sum of two functions is just the sum of the Laplace transform of the first function with that of the second. Using this fact and the tables of Laplace transforms, the control engineer can rather easily obtain the Laplace transform for many differential equations used in modeling physical systems. The advantage is that the resulting algebraic equation usually can be easily solved for the variables of interest. The *transfer function* of a control system is defined to be the ratio of the Laplace transform of the input variable to the Laplace transform of the output variable. By analyzing a control system in terms of the relation of the Laplace transform of the inputs to the Laplace transform of the outputs, it is possible to gain a good understanding of the system's performance properties.[2]

To make the analysis of control systems even easier, there are tables that provide the transfer functions for many of the differential equation relations encounted in control systems. An engineer can design a control system using various control components connected to one another by the way in which they pass signals. From these separate components, the engineer can derive the transfer function for the complete control system algebraically. The familiar block diagrams displayed in the control theory literature provide a convenient graphical representation of the underlying process model. The boxes in such diagrams are usually labeled with the transfer function for the corresponding system component and the arcs indicate the signals passing between components. Figure 2.5.i depicts the block diagram for a control system in which the output of the plant is fed back through some sort of a filter or amplifier and combined with the input to provide an error signal used by a compensator in controlling the plant. The control system pictured in Figure 2.5.i illustrates a simple instance of error-driven feedback, in which the system reference signal is continuously compared with the system's output in order to adjust various system parameters.[3]

Block diagrams can be simplified by algebraically combining the transfer functions of connected components according to a few simple rules [7]. For instance, the two blocks labeled $G_1(s)$ and $G_2(s)$ in Figure 2.5.i can be

[2]Frequency-domain methods involving transfer functions are so named because they allow the engineer to analyze the behavior of a system in terms of its response to inputs of varying frequencies and amplitudes. By evaluating the transfer function, $T(s)$, at $s = j\omega$ for any $\omega \in \mathbf{R}^+$, we obtain a complex number, $T(j\omega) = \alpha(\omega) + j\beta(\omega)$, whose magnitude, $\sqrt{\alpha^2(\omega) + \beta^2(\omega)}$, represents the response of the system in steady state to a sinusoidal input of frequency, ω, in terms of the ratio of the output to the input amplitude.

[3]In some texts, error-driven feedback is synonymous with *unity feedback*, corresponding to the case in which $H(s)$, in Figure 2.5.i, is the identity function.

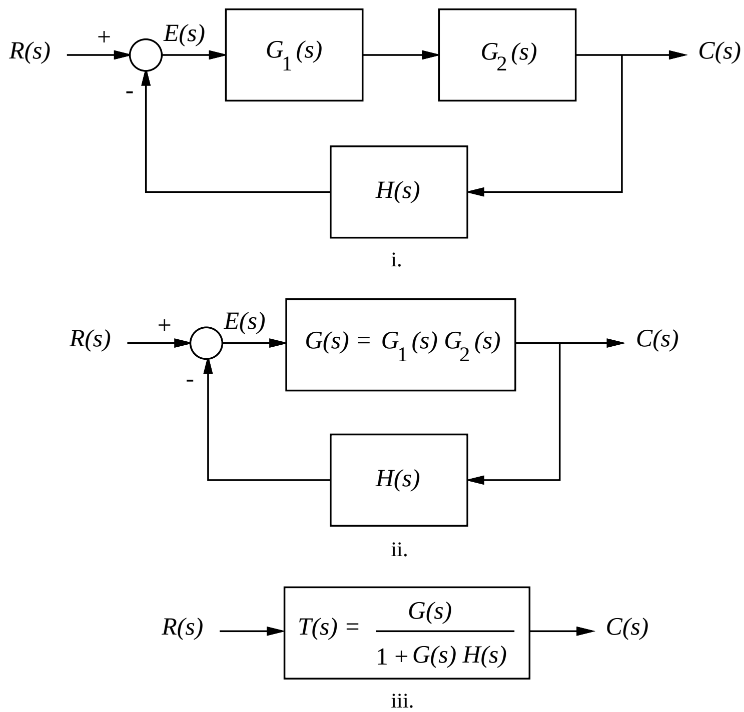

Figure 2.5: Block diagram of a control system utilizing feedback

combined to form,

$$G(s) = \frac{C(s)}{E(s)} = G_1(s)G_2(s),$$

noting that $C(s) = E(s)G_1(s)G_2(s)$. The simplified block diagram is shown in Figure 2.5.ii. The simplest block diagram is just a single box labeled with the transfer function for the complete control system. For instance, we can reduce the block diagram for the system shown in Figure 2.5.ii to a single component with input $R(s)$, output $C(s)$, and transfer function,

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)},$$

noting that $E(s) = R(s) - H(s)C(s)$ and $C(s) = E(s)G(s)$. This simplest block diagram is shown in Figure 2.5.iii. The function, $T(s)$, known as the *closed-loop transfer function*, is the basis of many existing control systems.

Much of the control theory found in textbooks deals with what are called *linear systems*. A system is said to be linear in terms of inputs and outputs if and only if it satisfies the properties of *superposition* and *homogeneity* [7]. A system satisfies the property of homogeneity if for any constant K and input x for which the output of the system is y, if the system is input Kx, the system outputs Ky. A system satisfies the superposition property if for any two inputs x_1 and x_2 with corresponding outputs y_1 and y_2, if the system is input $x_1 + x_2$, the system outputs $y_1 + y_2$. At first blush, the restriction to linear systems would seem to relegate much of control theory to a purely academic pursuit given that most natural systems are nonlinear at least in some range of their variables. Fortunately, we can develop reasonably accurate linear approximations by identifying almost-linear regions in the operating range of nonlinear systems. If the natural operating conditions of a system vary over a wide range, it may be necessary to develop several linear approximations and switch between them when necessary. This method of switching between controllers is the basis for a technique used in adaptive control called *gain scheduling*.

Other approximations are often made to simplify analysis and implementation. For instance, it is often possible to eliminate some of the higher-order terms in a model involving differential equations. By eliminating the higher-order terms, the subsequent analysis may ignore effects due to high-frequency inputs. Hopefully, these effects will not pose a problem in practice, but no model should be relied upon without careful experimentation comparing the performance of the modeled system with that of the real one.

While we have emphasized modeling continuous processes, control theory provides tools for modeling discrete processes as well. The discrete analog of a differential equation is called a *difference equation* and is used extensively not only to model discrete systems, but also to approximate continuous systems using digital hardware. Analog computers still play an important role in engineering, but, with the introduction of inexpensive digital computing hardware, a great deal of attention has been given to discrete modeling techniques.

Digital computers are limited in that they can only sample system variables at discrete points in time. Usually, the delay between samples is fixed of duration τ. By introducing a new complex variable, $z = e^{s\tau}$, we can define a discrete version of the Laplace transform called the *z-transform* for a discrete function $f(k)$ as

$$\mathcal{Z}(f(k)) = F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}.$$

There exist techniques, analogous to those based on the Laplace transform, for using the z-transform to analyze the response characteristics of control systems [3]. Analysis using the z-transform is complicated somewhat by the fact that information is irretrievably lost in a sampled system. It is generally necessary to identify the various frequency components of the input signal in the Fourier domain, and adjust the sampling rate accordingly to avoid effects due to *aliasing* (*i.e.,* mistakenly associating high frequency components of the signal with lower frequency components). It has been shown that aliasing can be avoided entirely by ensuring that the sampling frequency ($1/\tau$ samples per unit time) is at least twice the frequency of the highest frequency component of the input signal.[4] Of course, it may not be possible for the digital hardware to sample that quickly or perform the necessary computations required to generate an appropriate response. The problem of implementing complex control strategies that keep pace with a rapidly changing environment will be addressed frequently in this monograph.

There exist processes for which we know the form of an appropriate model (*e.g.,* we know that the process can be modeled using a kth-order linear differential equation with constant coefficients), but we do not know the parameters of the model. For instance, the system we are trying to model might be a black box that we know to be a single-input single-output linear system, but the model parameters do not correspond to any known physical parameters such as the spring constant or the damping constant in the model for the spring-mass-dashpot system. In this case, it may be possible to find values for the parameters of the model by sampling the input and output of the system, and "fitting" the parameters of the model to the data. This is a special case of what is called *system identification*, and constitutes an important part of the branch of control theory known as *adaptive control* [1, 13]. System identification can be done offline during the design of the control system as prologue to the sort of analysis described above. In adaptive control, system identification is done online by the control system, and the results of system identification are used to adjust the parameters of a controller. This approach to control is particularly useful if the physical system that you are attempting to model changes over time (*e.g.,* a plant with mechanical parts that are subject to wear).

[4]This result is generally attributed to Nyquist [11]. The frequency corresponding to twice the highest frequency component of the signal is called the *Nyquist rate*. A proof of this result by Shannon can be found in [14]. One consequence of Shannon's *sampling theorem* is that that the original signal can be recovered from the sampled data using so-called low-pass filters that pass low frequencies and attenuate higher frequencies.

One particularly convenient feature of the mathematical models used in control theory is that, at least as far as the analysis is concerned, what one learns about design in one area is immediately applicable in another area for which there exists appropriate analogical apparatus mapping the variables between the two systems [7]. For instance, the engineer familiar with the analysis and design of electrical control systems can often apply what he or she knows to the analysis and design of mechanical or fluid control systems. The basic models and their corresponding equations appear again and again, and hence much of what is learned can be compiled into tables, tools, and cookbook-style methods for dealing with commonly occurring specific cases [4].

In this section, we considered some of the basic techniques involved in modeling physical systems. We briefly touched upon some of the methods and terminology of control theory, specifically what is referred to as classical control theory. As was mentioned, classical control is most closely associated with analysis in the frequency domain. In the next section, we introduce a particular class of physical systems important from the standpoint of control, and consider modeling techniques drawn from *modern control theory*. Modern control theory is most closely associated with analysis in the time domain.

2.3 Modeling Dynamical Systems

The techniques described in the previous section are primarily useful for physical systems that can be modeled with a single input and a single output variable. In this section, we consider systems modeled with any finite number of input and output variables. We restrict our attention to a limited class of physical systems called *dynamical systems*. A dynamical system is defined by the following mathematical objects and axioms governing them.[5]

- A set of time points $T \subset \mathbf{R}$
- A set of states X
- A set of inputs U

[5]The definitions provided here roughly follow those of Kalman [10] though we have sacrificed rigor in some places to avoid lengthy technical commentary. Our objective here is to set the stage for a discussion of practical methods, and not, as in the case of Kalman's work, the precise description of mathematical abstractions.

- A set of outputs Y
- A set of input functions

$$\Sigma = \{\sigma : T \to U\}$$

- A state transition function

$$f : T \times T \times X \times U \to X$$

  whose value is the state $x(t) = f(t; \tau, x, \sigma) \in X$ resulting at time $t \in T$ starting from an initial state $x(\tau)$ at time $\tau \in T$ influenced by the action of the input σ.

- An output function

$$g : T \times X \to Y$$

We impose some additional restrictions. In particular, for any $t_1 < t_2 < t_3$ and $\sigma \in \Sigma$ we have

$$f(t_3; t_1, x, \sigma) = f(t_3; t_2, f(t_2; t_1, x, \sigma), \sigma),$$

and for any two input functions σ and σ' that agree on the interval $\langle t, \tau \rangle$ we have

$$f(t; \tau, x, \sigma) = f(t; \tau, x, \sigma').$$

The first of these restrictions provides a reasonable property that allows us to compose inputs. The second is often referred to as the *principle of causality* [2].[6] Given an input function $\sigma \in \Sigma$ and an interval of time $(t_1, t_2]$, an *input segment* $\sigma_{(t_1,t_2]}$ is just σ restricted to $(t_1, t_2]$. We require that, if $\sigma, \sigma' \in \Sigma$ and $t_1 < t_2 < t_3$, then there exists $\sigma'' \in \Sigma$ such that $\sigma''_{(t_1,t_2]} = \sigma_{(t_1,t_2]}$ and $\sigma''_{(t_2,t_3]} = \sigma_{(t_2,t_3]}$. This last property is called *concatenation of inputs* [10], and provides us with a useful closure property for the set of input functions.

We also assume that the response of a dynamical system is independent of the particular time at which it is exercised. We say that a dynamical system is *time invariant* if the following properties hold.

[6] There is a tendency in mathematical control theory to refer to certain assumptions or restrictions as principles. This is particularly the case where the mathematics would be difficult or impossible without imposing some restrictions. In some cases, such as the principle of causality described here, the restrictions seem innocuous enough, but in others they appear to be motivated by nothing more than mathematical convenience or necessity. Witness the fact that superposition, which underlies linearity, is often introduced as the "principle of superposition" [10].

- T is closed under addition.
- Σ is closed under the *shift operator*, $z^s : \sigma \mapsto \sigma'$, defined by

$$\sigma'(t) = \sigma(t + s)$$

  for all $s, t \in T$.
- For any $s, t, \tau \in T$, we have

$$f(t; \tau, x, \sigma) = f(t + s; \tau + s, x, z^s\sigma)$$

- The output function $g(t, .)$ is independent of t.

We will be concerned with *continuous time* dynamical systems (*i.e.,* T is the real numbers) and *discrete time* dynamical systems (*i.e.,* T is the integers). For mathematical purposes, we may introduce additional restrictions such as smoothness and linearity, but it should be pointed out that many physical systems cannot be modeled exactly under such restrictions.

We represent a continuous time-invariant dynamical system as

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t)) \end{aligned}$$

where the first equation is called the *state equation* and the second the *output equation*. The state and output equations typically consist of differential equations such that for any initial state $x(t_0)$ and input u both equations have unique solutions. The discrete counterpart of the continuous system is represented as

$$\begin{aligned} x(k + 1) &= f(x(k), u(k)) \\ y(k) &= g(x(k), u(k)) \end{aligned}$$

where the state equation in this case is a difference equation.

So far, we have treated states, inputs, and outputs as simple unstructured sets. Generally, the states, inputs, and outputs have considerable structure; it is often reasonable to represent each in terms of a multidimensional vector space (*e.g.,* $\mathbf{R}^n$). Each dimension of the space corresponds to a component *variable* of the corresponding vector space. For instance, in designing a dynamical system to model the fluid flow in and out of a holding tank, we might employ three state variables—the height of the fluid in the tank, the angle of the input valve, and the angle of the output valve. The resulting state space would be a subset of $\mathbf{R}^3$. In designing a system to model a robot, we might use the position in x, y, and z, and orientation in $\theta_{x,y}$, $\theta_{y,z}$, and $\theta_{x,z}$ for a six-dimensional state space, $\mathbf{R}^6$. In general, the state, input, or output variables may be boolean, real, integer, or discrete valued, and can correspond to any representable quantity or its derivatives, as long as the resulting space satisfies the requirements for being a finite-dimensional vector space [6]. By characterizing the states, inputs, and outputs in terms of linear vector spaces, we can bring to bear the considerable power of linear algebra and linear systems theory.

Much of linear control is concerned with linear time-invariant systems of the form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) \end{aligned}$$

where $\mathbf{x}$ is the n-dimensional state vector, $\mathbf{u}$ is the p-dimensional input vector, $\mathbf{y}$ is the q-dimensional output vector, and A, B, and C are, respectively, $n \times n$, $n \times p$, and $q \times n$ real constant matrices.

As a simple example illustrating how to construct a linear dynamical system, consider a single-degree-of-freedom robot of mass, M, acted upon by a force, $\mathcal{F}$. Let z be the position of the robot in some arbitrary frame of reference. We assume that the plane of motion is horizontal and that there are no frictional forces acting on the robot. The relationship between position, z, and the force, $\mathcal{F}$, is completely determined by Newton's second law of motion.

$$M\ddot{z} = \mathcal{F}$$

The dynamic behavior of the robot can be described in terms of the position and velocity of the robot, and, hence, we define the state vector to be,

$$\mathbf{x}(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}.$$

Equating the system state and the system output, we can write down the state and output equations,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} \mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{x}(t), \end{aligned}$$

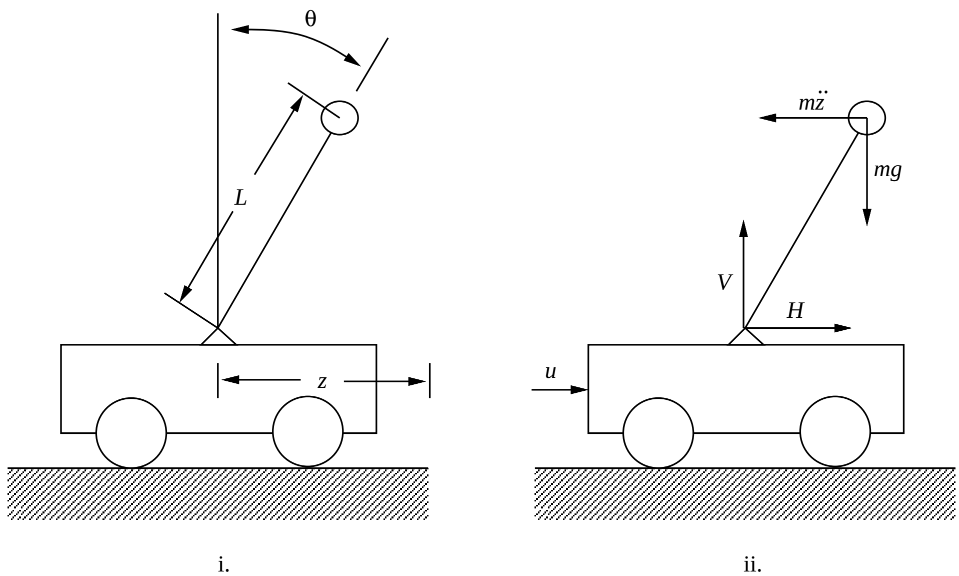

Figure 2.6: Inverted pendulum mounted on a cart

where, in this case, the input, $\mathbf{u}(t)$, is just a scalar-valued function of time. Generally, the system output contains incomplete information from which it is necessary to reconstruct the system state. In subsequent chapters, we consider some of the issues involved in attempting to infer the system state from incomplete information.

The restriction of linearity is a critical one that causes some researchers to dismiss much of mathematical control theory as a purely academic pursuit with no practical consequences. Most physical systems are nonlinear, and, hence, we can only approximate these systems using linear models. In many cases, such approximations are valid over only a limited range of the systems operating conditions. While these problems make it difficult to apply results from linear systems theory, the methods of linear systems theory are so powerful that the effort is often well spent. Nonlinear systems represent an important area of study in control theory [8], but a more detailed discussion is beyond the scope of this presentation.

To illustrate how to approximate a nonlinear system by a linear one, we consider a classic example in control that involves modeling an inverted pendulum mounted on a cart that can move back and forth along a horizontal track. This problem is often cited as an analogue of the problem of controlling a missile balanced atop its booster rockets [7, 9]. The presentation here follows that of Gopal [9]. We assume that the controller can exert a force on the cart to propel it to the right or left along the horizontal track. Let z be the horizontal position of the cart's center of gravity and $z + L\sin\theta$ the horizontal position of the center of gravity of the pendulum, where L is the distance from the pivot to the center of gravity of the pendulum. Similarly, $L\cos\theta$ is the vertical position of the center of gravity of the pendulum. Figure 2.6.i shows the basic configuration of cart and pendulum.

The state of the system is completely described by the position and velocity of the cart and the angular position and angular velocity of the pendulum. Thus we have the state vector:

$$\mathbf{x}(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix}.$$

In order to set up the dynamical equations, we have to establish some additional parameters. Let m be the mass of the pendulum, M the mass of the carriage, and J the moment of inertia of the pendulum with respect to its center of gravity.

The forces acting on the pendulum are the force of gravity, mg, acting on its center of gravity, a horizontal reaction force, H, and a vertical reaction force, V. Figure 2.6.ii depicts the forces acting on the pendulum and the cart. Taking moments about the center of gravity of the pendulum, we have

$$J\ddot{\theta}(t) = VL\sin\theta(t) - HL\cos\theta(t).$$

Summing all of the forces acting on the pendulum in the horizontal and vertical directions, we have

$$\begin{aligned} V - mg &= m\frac{d^2}{dt^2}(L\cos\theta(t)) \\ H &= m\frac{d^2}{dt^2}(z(t) + L\sin\theta(t)). \end{aligned}$$

Summing all of the forces acting on the cart, we have

$$u(t) - H = M\ddot{z}(t),$$

where $u(t)$ is the (control) input.

Since the task is to keep the pendulum upright, we will assume that θ and $\dot{\theta}$ will remain close to 0. On the basis of this assumption, we make the standard approximations, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, obtaining

$$\begin{aligned} mL\ddot{\theta}(t) + (m+M)\ddot{z}(t) &= u(t) \\ (J - mL^2)\ddot{\theta}(t) + mL\ddot{z}(t) - mgL\theta(t) &= 0 \end{aligned}$$

We introduce values for the remaining parameters.

$$M = 1 \text{ kg}, \quad m = 0.15 \text{ kg}, \quad L = 1 \text{ m}$$

Using any mechanics or physics textbook, we get

$$\begin{aligned} g &= 9.81 \text{ m/sec}^2 \\ J &= \frac{4}{3}mL^2 = 0.2 \text{ kg-m}^2 \end{aligned}$$

Using these equations and parameter values, we obtain

$$\begin{aligned} 0.15\,\ddot{\theta}(t) + 1.15\,\ddot{z}(t) &= u(t) \\ 0.35\,\ddot{\theta}(t) + 0.15\,\ddot{z}(t) - 0.15 \times 9.81\,\theta(t) &= 0 \end{aligned}$$

From these two equations, we solve for $\ddot{\theta}$ and $\ddot{z}$ and make appropriate substitutions, obtaining

$$\begin{aligned} \ddot{z}(t) &= -0.5809\,\theta + 0.9211\,u(t) \\ \ddot{\theta}(t) &= 4.4532\,\theta + -0.3947\,u(t) \end{aligned}$$

to arrive at the following state and output equations for the dynamical model:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -0.5809 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 4.4532 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0.9211 \\ 0 \\ -0.3947 \end{bmatrix} \mathbf{u}(t) \\ &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t) \\ &= C\mathbf{x}(t) \end{aligned}$$

where we assume realistically that the only component of the output that is directly observable is the angle, θ, corresponding to the tilt of the missile in the case of the booster rocket.

In Chapter 4, we highlight results from linear systems theory that allow us to establish important properties (*e.g.,* stability and controllability) of dynamical systems, using simple tests on the matrices that define the state and output equations. The inverted pendulum is particularly interesting as it represents a dynamical system that is not stable, but is controllable.

Before leaving this chapter, we introduce some additional concepts and terms. We will develop similar concepts in the next chapter, in some cases using the same terms and in other cases introducing new terminology. Where the terminology differs, we will point out the conceptual similarities. An *event* is simply a pair consisting of a time point and a state (*e.g.,* $\langle t, x \rangle$ where $t \in T$ and $x \in X$). The event (or *phase space*) space is the space of all possible events, $T \times X$.[7] A *state-space trajectory* is simply a mapping from the real interval to the state space, $h : [0, 1] \rightarrow X$, defined by a particular transition function, f, input, u, and initial conditions, $x(0) = x_0$. In the following chapter, we turn our attention to the use of logic in modeling physical systems.

2.4 Further Reading

Classical and modern control provide a wealth of modeling techniques. A basic familiarity with linear algebra and elementary differential equations are necessary for any real appreciation or application of these techniques. A good college-level physics course will also turn out to be useful in following the examples found in the introductory textbooks. Cannon [5] provides a wonderful introduction to modeling complex physical systems from an engineering standpoint.

For a general introduction to modeling from the perspective of automatic control, see the texts by Dorf [7] or Bollinger [3]. Dorf considers both modern and classical control in his text, but emphasizes the latter. For an emphasis on modern control theory, time-domain analysis, and, in particular, linear system theory, see Chen [6] or Gopal [9]. Our treatment of dynamical systems follows that of Kalman; Kalman's chapter in [10] provides a very general formulation of dynamical systems and an introduction to the necessary mathematical abstractions.

Control theory is a discipline steeped in mathematics. Some of the math-

[7] We follow Kalman [10] in our use of the term *phase space*. You may also see the term used to refer to the space of possible positions and velocities. A state variable obtained from a system variable and its derivative is referred to as a *phase variable* [9].

ematics is of a rather arcane sort, but much of it can be directly applied to solving real-world problems. While there is a growing body of work on nonlinear systems, a significant portion of the literature is devoted to a systematic exploitation of the properties of linear dynamic systems. In surveying that literature, it is interesting to note the wide range of physical phenomena for which such models are appropriate.

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