Iterated Linear Optimization

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Overview

Discrete dynamical system defined by repeated linear optimization.

Compact convex set $\Delta \subseteq \mathbb{R}^n$, $T : \Delta \to \Delta$,

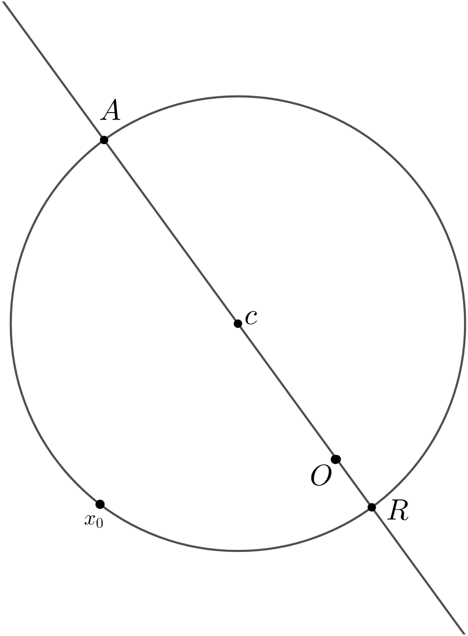
$$T(x) = \underset{y \in \Delta}{\operatorname{argmax}}(x \cdot y)$$

Fixed point iteration generates a sequence $\{x_0, x_1, x_2, \ldots\}$ where

$$x_{t+1} = T(x_t)$$

- Fixed points reflect geometric properties of Δ.
- Can be used for rounding solutions of semidefinite relaxations.
- We characterize the fixed points in *elliptopes*.

Example 1



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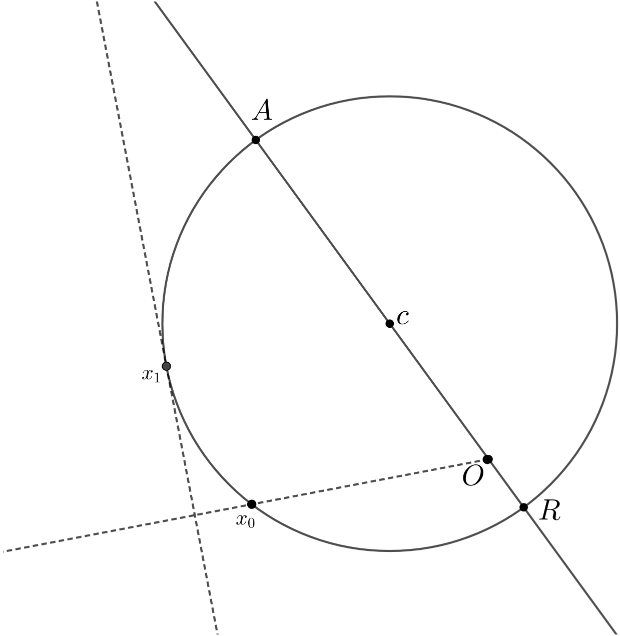

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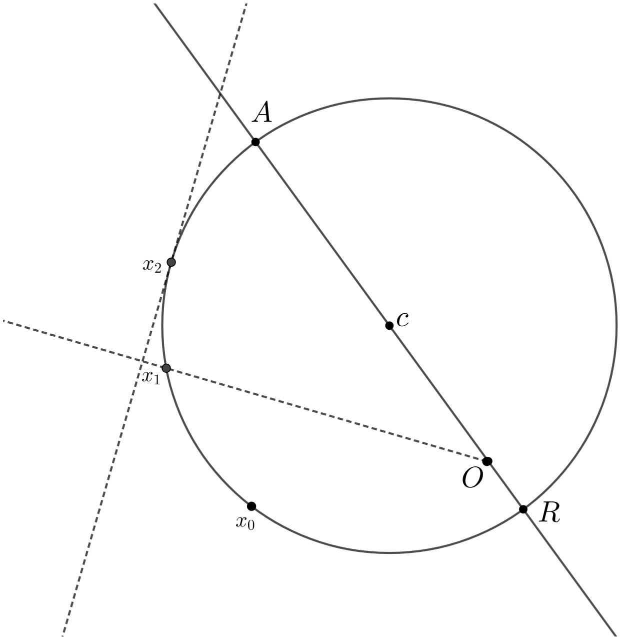

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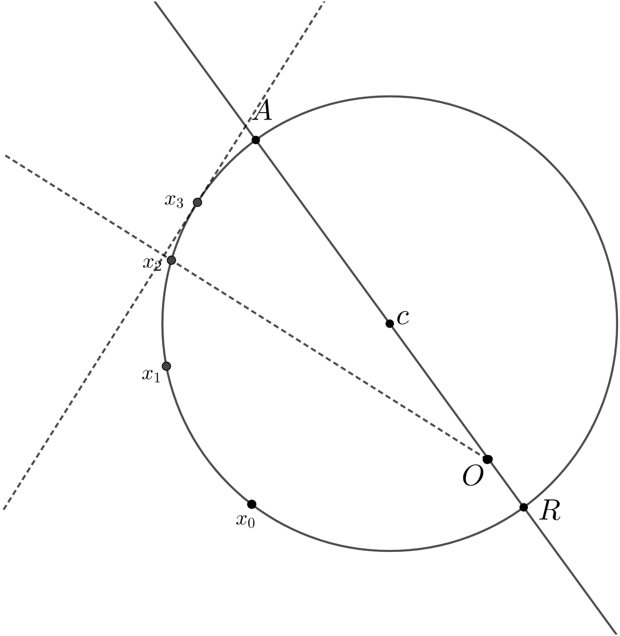

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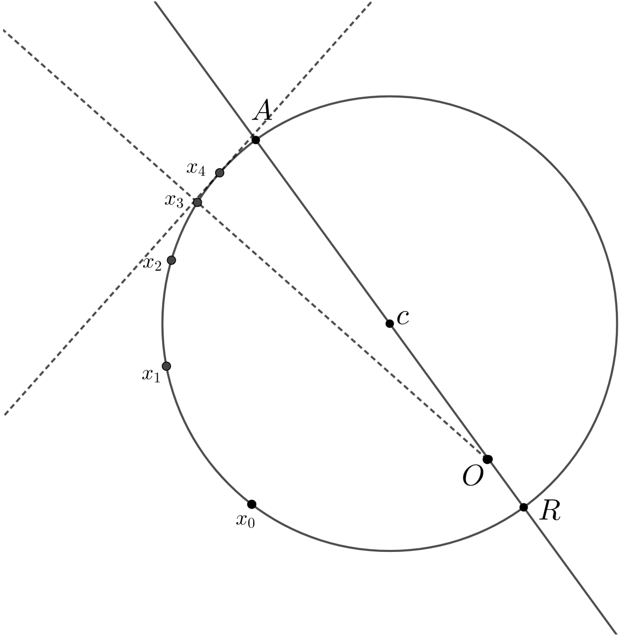

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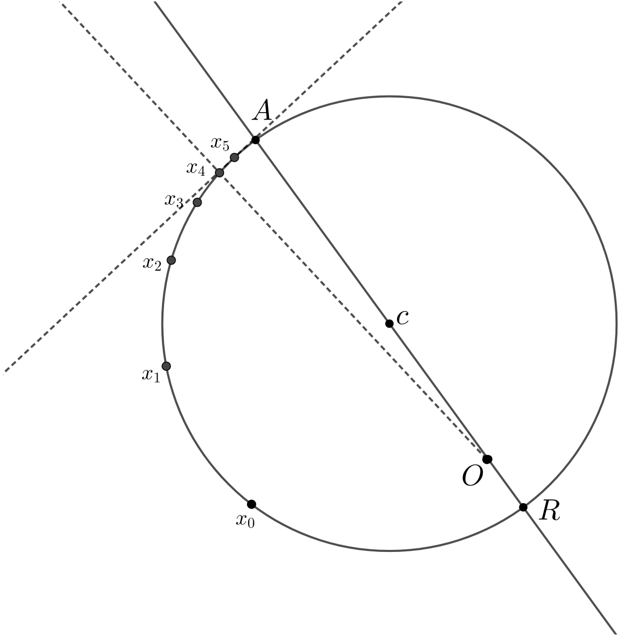

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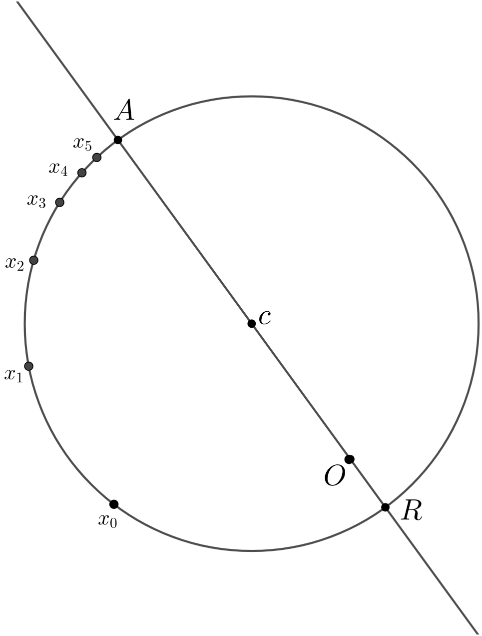
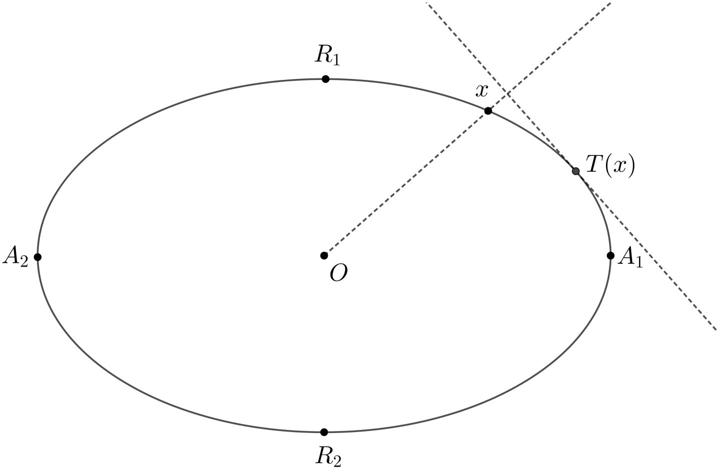


Example 2



Fixed point iteration

Theorem (FKP20)

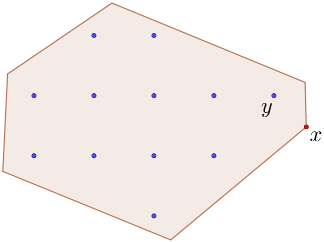
Sequence $\{x_0, x_1, x_2, \ldots\}$ *converges to a fixed point of* T.

Iterative method for maximizing $||x||^2$ in Δ.

Related to Franke-Wolfe optimization method.

Convex relaxations

$$\underset{z \in S}{\operatorname{argmax}} f(z)$$



Combinatiorial optimization via convex relaxation:

1. Discrete set of possible solutions S relaxed to convex set Δ.
2. Optimize objective over Δ.
3. "Round" solution $x \in \Delta$ to solution $y \in S$.

Iteration with T can be used for rounding semidefinite relaxations.

Elliptope

$\mathcal{S}_n = n$ by n symmetric matrices.

Elliptope $\mathcal{L}_n \subseteq \mathcal{S}_n$ are positive semidefinite matrices with 1's on diagonal:

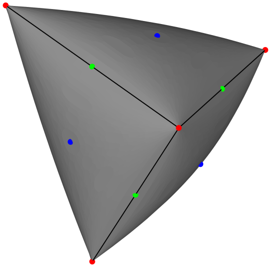
$$\mathcal{L}_n = \{X \in \mathcal{S}_n \,|\, X \succcurlyeq 0, X_{i,i} = 1\}.$$

- Goemans-Williamson semidefinite relaxation for max-cut.
- Gram matrices of n unit vectors in $\mathbb{R}^n$.
- Instance of *spectrahedron*.

Fixed points in $\mathcal{L}_n$

$\mathcal{L}_3$ can be visualized in $\mathbb{R}^3$.

$$X = \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix}$$



Red fixed points are irreducible matrices with rank 1.
Blue fixed points are irreducible matrices with rank 2.
Green fixed points are reducible matrices with rank 2.

Algebraic Characterization

Lemma (FKP20)

If $X = T(M)$ there exists a diagonal matrix D such that

$$MX = DX.$$

(similar to eigenvector $Mx = \lambda x$)

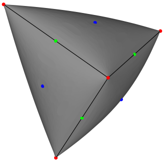
Theorem (FKP20)

$X = T(X)$ iff there exists a diagonal matrix D such that

$$X^2 = DX.$$

Elliptopes

$\mathcal{L}_3$: finite number of fixed points.



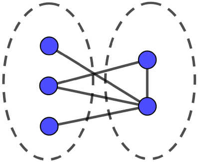
$\mathcal{L}_4$: inifinite number of fixed points.
Any $-1 < c < 1$ leads to a distinct fixed point:

$$X(c) = \begin{pmatrix} 1 & -\sqrt{1-c^2} & 0 & c \\ -\sqrt{1-c^2} & 1 & -c & 0 \\ 0 & -c & 1 & -\sqrt{1-c^2} \\ c & 0 & -\sqrt{1-c^2} & 1 \end{pmatrix}$$

$\mathcal{L}_n$: finite number of *regular* fixed points.
(one-dimensional normal cone)

Rounding max-cut relaxation

Max-cut:
partition vertices of graph maximizing
the weight of edges between sets.



Semidefinite relaxation involves linear optimization over $\mathcal{L}_n$.

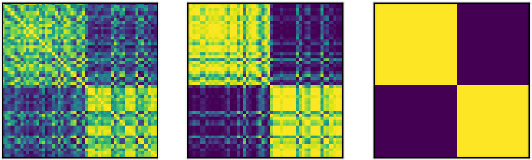
Partitions of $\{1, \ldots, n\}$ are the vertices of $\mathcal{L}_n$.

Round $X \in \mathcal{L}_n$ by finding the closest vertex Y.

- Relax to $\mathcal{L}_n$: Y=T(X).
- If Y is *not* a vertex, we iterate to find vertex close to Y.

The vertices of $\mathcal{L}_n$ are the attractive fixed points of T.

Rounding max-cut relaxation



x_0 x_1 x_3

Fixed point iteration starting from the solution of the max-cut relaxation for a graph with 50 vertices and random weights.