FORMULATIONS AND GENERALIZATIONS OF EISENSTEIN SERIES

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1. Introduction

A newcomer to the study of modular and automorphic forms will certainly come across the following definition of the Eisenstein series:

$$G_k(z) := \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^k}. \tag{1.1}$$

Here and throughout, we take $z \in \mathbb{H}$, where $\mathbb{H} := \{x + iy \in \mathbb{C} : y > 0\}$ is the upper-half of the complex plane. Various generalizations of this series appear, such as

$$E_{k,\mathfrak{a}}(z) := \sum_{\gamma\in\Gamma_\mathfrak{a}\setminus\Gamma} j_{\sigma_\mathfrak{a}^{-1}\gamma}(z)^{-k}. \tag{1.2}$$

The purpose of this expository note is to bridge this conceptual gap. For the three variants of Eisenstein series listed below, we present a definition, verify that it's well-defined and absolutely convergent, and show that it satisfies the relevant modular transformation law. The Eisenstein series we will consider are, in increasing orders of sophistication:

(1) the Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$, defined in (1.1) as a series over the $\mathbb{Z}^2$ lattice;
(2) the normalized Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$, defined as an average over the quotient $\Gamma_\infty\setminus\mathrm{SL}_2(\mathbb{Z})$; and
(3) the Eisenstein series for $\Gamma = \Gamma_0(q)$ at the cusp $\mathfrak{a}$, defined in (1.2) as an average over the quotient $\Gamma_\mathfrak{a}\setminus\Gamma$.

When possible, we illustrate how the problem of understanding the Eisenstein series at one level of generality can be reduced to understanding those Eisenstein series at the previous level. Throughout this note, we use freely basic facts and notations relevant to the theory of modular forms; good sources for these are [1, 3].

2. Eisenstein series as an average over $\mathbb{Z}^2$

Here we present the first Eisenstein series that one typically encounters. It has an elementary definition:

Definition 2.1. *Fix $k \in \mathbb{Z}$. For $z \in \mathbb{H}$, the weight-k Eisenstein series is*

$$G_k(z) := \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^k}. \tag{2.1}$$

Lemma 2.2. *If $k > 2$, then the following are true:*

(1) *$G_k(z)$ converges absolutely;*
(2) *$G_k(z)$ converges uniformly on compact sets, and is therefore holomorphic on $\mathbb{H}$;*

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(3) *For every* $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, *we have* $G_k(\gamma z) = (cz+d)^k G_k(z)$.

(4) *If* k *is odd, then* $G_k(z) = 0$ *identically.*

Proof. Towards proving (1), we'll first argue that for every $z \in \mathbb{H}$, there exists $\delta = \delta(z) \in (0,1)$ with

$$|mz+n| \geq \delta |mi+n|, \tag{2.2}$$

for all $(m,n) \neq (0,0)$. If $m=0$, then this is true for any $\delta \in (0,1)$. And if $m \neq 0$, then this is equivalent to

$$\left| \frac{z+n/m}{i+n/m} \right| \geq \delta.$$

Towards finding a suitable δ, we consider the function $f_z : \mathbb{R} \to \mathbb{R}$ defined by $f_z(x) = |(z-x)/(i-x)|$. It's clear that f_z is continuous, that $f_z(x) > 0$ for all $x \in \mathbb{R}$, and that $f_z(x) \to 1$ as $x \to \pm\infty$. Hence, there exists an $R \in \mathbb{R}$ such that $f_z(x) \geq 1/2$ for all $|x| \geq R$. And for $x \in [-R,R]$, positivity of $f_z(x)$ implies (by compactness) that there exists $c > 0$ such that $f_z(x) \geq c$. It follows that, for all $x \in \mathbb{R}$, we have $f_z(x) \geq \delta = \min\{1/2, c\}$, so (2.2) indeed holds. This lets us estimate:

$$\sum_{(m,n)\neq(0,0)} \frac{1}{|mz+n|^k} \leq \sum_{(m,n)\neq(0,0)} \frac{1}{\delta^k |mi+n|^k} \ll_z \sum_{(m,n)\neq(0,0)} \frac{1}{(\sqrt{m^2+n^2})^k}.$$

This sum converges by Lemma 2.3 below, which also justifies our restriction $k > 2$. It follows that $G_k(z)$ converges absolutely.[1] It's clear that a suitable δ can be chosen uniformly for z in a compact subset of $\mathbb{H}$, so (2) follows.

It's a standard result that the Möbius transformations

$$z \mapsto z+1, \qquad z \mapsto -1/z \tag{2.3}$$

generate the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ (c.f. [1]). One can check that the transformation law $G_k(\gamma z) = (cz+d)^k G_k(z)$ holds for general $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ if it holds for these distinguished transformations (2.3). (The elementary inductive proof of this fact uses the chain rule $j_{\sigma\tau}(z) = j_\sigma(\tau z) j_\tau(z)$, where $j_\gamma(z) := cz+d$.) But the transformation rule (3) under the generators (2.3) holds because absolute convergence of $G_k(z)$ permits us to rearrange at will: for the former transformation, we compute

$$G_k(z+1) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+(m+n))^k} = G_k(z),$$

and for the latter,

$$G_k(-1/z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m(-1/z)+n)^k} = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{(-z)^k}{(m-nz)^k} = (-z)^k G_k(z).$$

To see that (4) holds, if we apply (3) with $\gamma = -I$, then we have $f(-Iz) = (-1)^k f(z)$, so if k is odd then $f = 0$. □

[1] We credit this proof of absolute convergence, as well as Lemma 2.3, to Keith Conrad's notes [2].

Lemma 2.3. *For $(m,n) \in \mathbb{Z}^2$, if we define $\|(m,n)\| := \sqrt{m^2+n^2}$, then the lattice series*

$$\sum_{a \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{\|a\|^s} \tag{2.4}$$

diverges for $0 < s \leq 2$ and converges for $s > 2$.

Proof. In (2.4), we first collect terms of the same size. If we define

$$r_2(n) := \left\{(a_1, a_2) \in \mathbb{Z}^2 : a_1^2 + a_2^2 = n\right\},$$

then we can compute

$$\sum_{a \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{\|a\|^s} = \sum_{a \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(\|a\|^2)^{s/2}} = \sum_{n \geq 1} \frac{r_2(n)}{n^{s/2}} = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{r_2(n)}{n^{s/2}}.$$

In order to estimate this limiting behavior, we'll use the summation by parts formula

$$\sum_{n=1}^{N} u_n(v_n - v_{n-1}) = u_N v_N - u_1 v_0 - \sum_{n=1}^{N-1} v_n(u_{n+1} - u_n).$$

(Note the similarity to the formula $\int u\, dv = uv - \int v\, du$ for integration by parts.) If we define $S(n) := r_2(1) + \cdots + r_2(n)$ and $S(0) = 0$, then we have

$$\sum_{n=1}^{N} \frac{r_2(n)}{n^{s/2}} = \sum_{n=1}^{N} \frac{S(n) - S(n-1)}{n^{s/2}},$$

so taking $u_n = 1/n^{s/2}$ and $v_n = S(n)$, we deduce that

$$\sum_{n=1}^{N} \frac{S(n) - S(n-1)}{n^{s/2}} = \frac{S(N)}{N^{s/2}} - \sum_{n=1}^{N-1} S(n) \left(\frac{1}{(n+1)^{s/2}} - \frac{1}{n^{s/2}} \right).$$

We can rewrite the difference on the RHS as an integral using the fundamental theorem of calculus:

$$\frac{1}{(n+1)^{s/2}} - \frac{1}{n^{s/2}} = \int_n^{n+1} \frac{d}{dx} \left(\frac{1}{x^{s/2}} \right) dx = -\frac{s}{2} \int_n^{n+1} \frac{1}{x^{s/2+1}} dx.$$

This yields

$$\sum_{n=1}^{N} \frac{S(n) - S(n-1)}{n^{s/2}} = \frac{S(N)}{N^{s/2}} + \frac{s}{2} \sum_{n=1}^{N-1} \int_n^{n+1} \frac{S(n)}{x^{s/2+1}} dx = \frac{S(N)}{N^{s/2}} + \frac{s}{2} \int_1^N \frac{S(x)}{x^{s/2+1}} dx,$$

if we define $S(x) := S(\lfloor x \rfloor)$.

Next, we estimate $S(N)$ geometrically (c.f. the Gauss circle problem). The quantity $S(x)$ counts nonzero integral points inside the circle of radius $\sqrt{x}$:

$$S(x) = \#\left\{a \in \mathbb{Z}^2 \setminus \{(0,0)\} : \|a\| \leq \sqrt{x}\right\}.$$

This quantity is approximately the area of the ball of radius $\sqrt{x}$. More precisely, for large x, this quantity is at least the area of the ball of radius $\sqrt{x} - 2$, and it is at most the area of the ball of radius $\sqrt{x} + 2$. That is, for large enough x,

$$\pi(\sqrt{x} - 2)^2 \leq S(x) \leq \pi(\sqrt{x} + 2)^2.$$

As $\sqrt{x} \pm 2 \asymp \sqrt{x}$, there exist absolute constants such that $A \cdot x \leq S(x) \leq B \cdot x$ for large enough x. This implies that, for some absolute constant $M > 0$, we have

$$\frac{Ax}{x^{s/2+1}} \leq \frac{S(x)}{x^{s/2+1}} \leq \frac{Bx}{x^{s/2+1}} \tag{2.5}$$

whenever $x \geq M$. We're now in a position to directly analyze the limit

$$\sum_{a \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{\|a\|^s} = \lim_{N \to \infty} \left(\frac{S(N)}{N^{s/2}} + \frac{s}{2} \int_1^N \frac{S(x)}{x^{s/2+1}} dx \right).$$

If $0 < s \leq 2$, then $s/2 + 1 \leq 2$, so $Ax/x^{s/2+1} \geq A/x$, so by (2.5) we have

$$\int_1^N \frac{S(x)}{x^{s/2+1}} dx \geq \int_M^N \frac{A}{x} dx \to \infty$$

as $N \to \infty$, so the lattice series diverges. But if $s > 2$, then $s/2 + 1 \geq 2 + \epsilon$ for some $\epsilon > 0$, so $Bx/x^{s/2+1} \leq B/x^{1+\epsilon}$, so by (2.5) we have

$$\int_1^N \frac{S(x)}{x^{s/2+1}} dx \leq \int_1^M \frac{S(x)}{x^{s/2+1}} dx + \int_M^N \frac{B}{x^{1+\epsilon}} dx,$$

hence the integral converges as $N \to \infty$. Furthermore, when $s > 2$ we have $S(N)/N^{s/2} \to 0$, since (2.5) gives

$$\frac{S(N)}{N^{s/2}} \leq \frac{B}{N^{s/2-1}} \leq \frac{B}{N^{\epsilon}} \to 0.$$

This completes the proof. □

3. Eisenstein series as an average over $\mathrm{SL}_2(\mathbb{Z})$

To define this Eisenstein series, we need the following distinguished subgroup of $\mathrm{SL}_2(\mathbb{Z})$:

$$\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} =: \{\pm T^n : n \in \mathbb{Z}\}.$$

This notation will be explained in the following section. We also need the automorphy factor

$$j_\gamma(z) := cz + d,$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Definition 3.1. *Fix $k \in \mathbb{Z}$. For $z \in \mathbb{H}$, the weight-k normalized Eisenstein series is*

$$E_k(z) := \sum_{\gamma \in \Gamma_\infty \setminus \mathrm{SL}_2(\mathbb{Z})} j_\gamma(z)^{-k}. \tag{3.1}$$

Lemma 3.2. *If $k > 2$ is even, then the following are true:*

1. *The summands in (3.1) are well-defined; and*
2. *$E_k(z) = G_k(z)/2\zeta(k)$.*

In particular, for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have $E_k(\gamma z) = j_\gamma(z)^k E_k(z)$.

Proof. Towards verifying that the summands are well-defined, we'll first show that following map is a (well-defined) bijection:

$$\Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z}) \to \big\{(c,d) \in \mathbb{Z}^2 : c > 0, (c,d) = 1\big\} \cup \{(0,1)\} \tag{3.2}$$
$$\Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{cases} (0,1) & c = 0 \\ (|c|, \operatorname{sgn}(c) \cdot d) & c \neq 0. \end{cases}$$

Write $\Gamma_\infty \gamma_1 = \Gamma_\infty \gamma_2$, where

$$\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}. \tag{3.3}$$

If $\gamma_1, \gamma_2 \in \Gamma_\infty$, then $c_1 = c_2 = 0$, so clearly (3.2) is well-defined in this case. Now assume that $\gamma_1, \gamma_2 \notin \Gamma_\infty$. There are two cases to consider: if $\gamma_1 = T^n \gamma_2$, then we compute

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} * & * \\ c_2 & d_2 \end{pmatrix},$$

so clearly $|c_1| = |c_2|$ and $\operatorname{sgn}(c_1)d_1 = \operatorname{sgn}(c_2)d_2$; and if $\gamma_1 = -T^n\gamma_2$, then

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} * & * \\ -c_2 & -d_2 \end{pmatrix},$$

so again $|c_1| = |c_2|$ and $\operatorname{sgn}(c_1)d_1 = \operatorname{sgn}(c_2)d_2$. It follows that (3.2) is well-defined.

To check injectivity, let us first assume that γ_1 and γ_2, given by (3.3), satisfy $c_1 = c_2 = 0$; then clearly $\gamma_1, \gamma_2 \in \Gamma_\infty$, so $\Gamma_\infty\gamma_1 = \Gamma_\infty\gamma_2$, as needed. Next let us assume γ_1 and γ_2 yield $(|c_1|, \operatorname{sgn}(c_1) \cdot d_1) = (|c_2|, \operatorname{sgn}(c_2) \cdot d_2)$. Since (3.2) is well-defined, we may assume each $c_i \geq 0$, as $-I \in \Gamma_\infty$. This implies $c_1 = c_2$ and $d_1 = d_2$, so we can compute

$$\gamma_1\gamma_2^{-1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} d_2 & -b_2 \\ -c_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1d_1 - b_1c_1 & * \\ c_1d_1 - c_1d_1 & a_2d_2 - b_2c_2 \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix},$$

which is in Γ_∞. This implies that $\Gamma_\infty\gamma_1 = \Gamma_\infty\gamma_2$, so (3.2) is indeed injective. And (3.2) is surjective, because any $\gamma \in \Gamma_\infty$ satisfies $c = 0$, and given $(c,d) = 1$ with $c > 0$, there exist $a, b \in \mathbb{Z}$ with $ad + (-b)c = 1$.

Now, we'll argue that (3.2) is well-defined implies that different coset representatives $\Gamma_\infty\gamma_1 = \Gamma_\infty\gamma_2$ yield the same summands[2] in (3.1). Assume $\Gamma_\infty\gamma_1 = \Gamma_\infty\gamma_2$ with γ_i as in (3.3). If $\gamma_1, \gamma_2 \in \Gamma_\infty$, then the automorphy factor of each γ_i is $(-1)^k$; these are equal so long as k is even. Now suppose $\gamma_1, \gamma_2 \notin \Gamma_\infty$, so $(|c_1|, \operatorname{sgn}(c_1) \cdot d_1) = (|c_2|, \operatorname{sgn}(c_2) \cdot d_2)$. If $c_1 = c_2$, then $d_1 = d_2$, which implies $c_1z + d_1 = c_2z + d_2$, so of course $(c_1z + d_1)^k = (c_2z + d_2)^k$; and if $c_1 = -c_2$, then $d_1 = -d_2$, so

$$(c_1z + d_1)^k = (-c_2z - d_2)^k = (c_2z + d_2)^k$$

because k is even. This proves (1).

Furthermore, that (3.2) is a bijection allows us to rewrite E_k in the following form:

$$E_k(z) = 1 + \sum_{\substack{c>0 \\ (c,d)=1}} \frac{1}{(cz+d)^k}. \tag{3.4}$$

[2]In general, they don't yield the same automorphy factor because of the sign issue that we take care of in (3.2). For example, $j_I(z) = 1$ and $j_{-I}(z) = -1$, even though $\Gamma_\infty I = \Gamma_\infty(-I)$. This is not a problem, since we raise this automorphy factor to the even power k.

(The constant term comes from the $c = 0, d = 1$ summand.) Such a re-indexing is justified, provided we can show $E_k(z)$, in this new form, converges absolutely. But this converges absolutely by direct comparison to the Eisenstein series $G_k(z)$: the normalized Eisenstein series $E_k(z)$ is a series whose summands are a subset of the summands of $G_k(z)$.

Next, we can rearrange the series defining $G_k(z)$, gathering terms according to the greatest common divisor of m and n, yielding

$$G_k(z) = \sum_{d\geq 1} \sum_{(m,n)=d} \frac{1}{(mz+n)^k} = \sum_{d\geq 1} \frac{1}{d^k} \sum_{(m,n)=d} \frac{1}{((m/d)z + (n/d))^k}.$$

The sum over $(m,n) = d$ is equivalent to a sum over relatively prime m/d and n/d, hence

$$G_k(z) = \zeta(k) \sum_{(m',n')=1} \frac{1}{(m'z+n')^k}.$$

Therefore to prove (2), it remains to show that

$$1 + 1 \sum_{\substack{c>0 \\ (c,d)=1}} \frac{1}{(cz+d)^k} = \frac{1}{2} \sum_{(m',n')=1} \frac{1}{(m'z+n')^k}.$$

The involution $(c,d) \mapsto (c,-d)$ on the LHS summands covers all the RHS summands, save for $(m',n') = (0, \pm 1)$, which corresponds to the constant term on the LHS. $\square$

We showed that $E_k(z)$ satisfies the modular transformation rule by verifying that $E_k(z)$ is a constant multiple of $G_k(z)$, and therefore inherits the transformation law from $G_k(z)$. But this is "morally wrong." Since $E_k(z)$ was defined to be an average over the action of the modular group, the transformation law should follow directly from the definition (once we are allowed to rearrange the summands.) Namely, we can compute that

$$E_k(\gamma z) = \sum_{\tau \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} j_\tau(\gamma z)^{-k} = j_\gamma(z)^k \sum_{\tau \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} j_{\tau\gamma}(z)^{-k} = j_\gamma(z)^k E_k(z).$$

This simple proof uses only absolute convergence to justify the rearrangement, as well as the chain rule for j, which says that

$$j_{\tau\gamma}(z) = j_\tau(\gamma z) j_\gamma(z). \tag{3.5}$$

4. Eisenstein series as an average over $\Gamma_0(q)$, at the cusp $\mathfrak{a}$

We let k be an integer, we fix $\mathfrak{a}$ a cusp of $\Gamma := \Gamma_0(q)$, and denote by $\Gamma_{\mathfrak{a}}$ the stabilizer of $\mathfrak{a}$ inside Γ. Let $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{Z})$ be the scaling matrix which maps the cusp ∞ to the cusp $\mathfrak{a}$. See [3] for definitions of all these.

Definition 4.1. *For $z \in \mathbb{H}$, the weight-k Eisenstein series on $\Gamma_0(q)$ at the cusp $\mathfrak{a}$ is*

$$E_{\mathfrak{a}}(z) := E_{k,\mathfrak{a}}(z) := \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-k}. \tag{4.1}$$

For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\mathfrak{a} = \infty$, we have $\sigma_\infty = I$, which means $E_\infty(z)$ is just the normalized Eisenstein series of weight k defined in (3.1).

Lemma 4.2. *If $k > 2$ is even, then the following are true:*

1. *The summands in (4.1) are well-defined.*
2. *$E_{\mathfrak{a}}(z)$ converges absolutely in $\mathbb{H}$.*

(3) *For any* $\gamma \in \Gamma$, *we have* $E_{\mathfrak{a}}(\gamma z) = j_\gamma(z)^k E_{\mathfrak{a}}(z)$.

Proof. We'll first argue the summands are well-defined. To bootstrap our previous results, it's prudent to transform the series (4.1) into one where the average is taken over the quotient of some matrix group modulo Γ_∞. Such a transformation is furnished by the following correspondence:

$$\left\{\sigma_{\mathfrak{a}}^{-1}\gamma : \Gamma_{\mathfrak{a}}\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma\right\} = \left\{\gamma\sigma_{\mathfrak{a}}^{-1} : \Gamma_\infty\gamma \in \Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}\right\}. \tag{4.2}$$

(Proof: if $\gamma \in \Gamma$, then $\sigma_{\mathfrak{a}}^{-1}\gamma = (\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}})\sigma_{\mathfrak{a}}^{-1} \in$ RHS; conversely, if $\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$, then $(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}})\sigma_{\mathfrak{a}}^{-1} \in$ LHS.) This one-to-one correspondence is well-defined up to choice of coset representative, so to show well-definedness of the summands in (4.1), it suffices to show well-definedness of the summands in

$$E_{\mathfrak{a}}(z) = \sum_{\gamma \in \Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}} j_{\gamma\sigma_{\mathfrak{a}}^{-1}}(z)^{-k} = j_{\sigma_{\mathfrak{a}}^{-1}}(z)^{-k} \sum_{\gamma \in \Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}} j_\gamma(\sigma_{\mathfrak{a}}^{-1}z)^{-k}, \tag{4.3}$$

where the second equality follows from the chain rule (3.5). That the RHS series has well-defined summands follows directly from well-definedness of the summands in the weight-k normalized Eisenstein series (3.1), which proves (1). This also proves (2), since (4.3) converges absolutely by comparison to the weight-k normalized Eisenstein series (3.1).

It remains to verify that $E_{\mathfrak{a}}$ satisfies the modular transformation law (3). We compute, for any $\tau \in \Gamma$, that

$$E_{\mathfrak{a}}(\tau z) = \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(\tau z)^{-k} = j_\tau(z)^k \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} j_{\sigma_{\mathfrak{a}}^{-1}\gamma\tau}(z)^{-k} = j_\tau(z)^k E_{\mathfrak{a}}(z)$$

by rearranging the series. ☐

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