A binary relation R is **reflexive** if for all $a \in A$, aRa. It is **symmetric** if for all $a, b \in A$, aRb if and only if bRa. It is **transitive** if for all $a, b, c \in A$, if aRb and bRc, then aRc.

A binary relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive. For example, the pairs (a, b), $a, b \in \mathbb{N}$, for which both a and b have the same remainder on division by 3, is an equivalence relation. (See Problem 1.3.)

If R is an equivalence relation and aRb, then a and b are said to be **equivalent elements**. We let $E[a]$ be the set of elements in A that are equivalent to a under the relation R and call it the **equivalence class** of elements equivalent to a. It is not difficult to show that for all $a, b \in A$, $E[a]$ and $E[b]$ are either equal or disjoint. (See Problem 1.4.) Thus, the equivalence classes of an equivalence relation over a set A partition the elements of A into disjoint sets. For example, the partition $\{0^*, 0(0^*10^*)^+, 1(0+1)^*\}$ of the set $(0+1)^*$ of binary strings defines an equivalence relation R. The equivalence classes consist of strings containing zero or more 0's, strings starting with 0 and containing at least one 1, and strings beginning with 1. It follows that $00R000$ and $1001R11$ hold but not $10R01$.

1.2.5 Graphs

A **directed graph** $G = (V, E)$ consists of a finite set V of distinct **vertices** and a finite set of pairs of vertices $E \subseteq V \times V$ called **edges**. **Edge** e **is incident on vertex** v if e contains v. A directed graph is **undirected** if for each edge (v_1, v_2) in E the edge (v_2, v_1) is also in E. Figure 1.2 shows two examples of directed graphs, some of whose vertices are labeled with symbols denoting gates, a topic discussed in Section 1.2.7. In a directed graph the edge (v_1, v_2) is directed from the vertex v_1 to the vertex v_2, shown with an arrow from v_1 to v_2. The **in-degree** of a vertex in a directed graph is the number of edges directed into it; its **out-degree** is the number of edges directed away from it; its **degree** is the sum of its in- and out-degree. In a directed graph an **input vertex** has in-degree zero, whereas an **output vertex** either has out-degree zero or is simply any vertex specially designated as an output vertex. A **walk** in a graph (directed or undirected) is a tuple of vertices $(v_1, v_2, \ldots, v_p)$ with the property that (v_i, v_{i+1}) is in E for $1 \leq i \leq p-1$. A walk $(v_1, v_2, \ldots, v_p)$ is **closed** if $v_1 = v_p$. A **path** is a walk with distinct vertices. A **cycle** is a closed walk with $p-1$ distinct vertices, $p \geq 3$. The **length of a path** is the number of edges on the path. Thus, the path $(v_1, v_2, \ldots, v_p)$ has length $p-1$. A **directed acyclic graph (DAG)** is a directed graph that has no cycles.

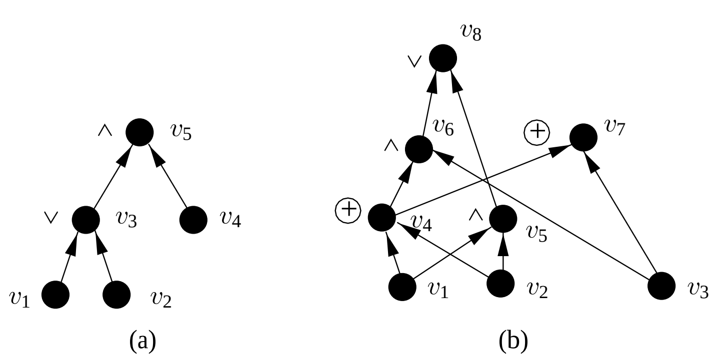

Figure 1.2 Two directed acyclic graphs representing logic circuits.