Figuring out the amplitude of the sun – an explanation of what the spreadsheet is doing.

The amplitude of the sun (at sunrise, say) is the direction you look to see the sun come up. If it's rising exactly in the east, its amplitude is 90E, which is usually written in the form of an azimuth: N 90 E. If it's in the southeast, maybe it's N 105 E. If it's setting a little north of west, it's N 80 W. (i.e., start at north, and go 80 degrees towards the west).

The model used for computing this assumes that

- The earth's in a circular orbit (not elliptical) about the sun, and moves with constant speed.
- The earth's axis is inclined at a constant angle (23.5 degrees) to the plane of its orbit
- This inclined axis lies in the plane perpendicular to the plane of the orbit and containing the earth at the winter equinox.
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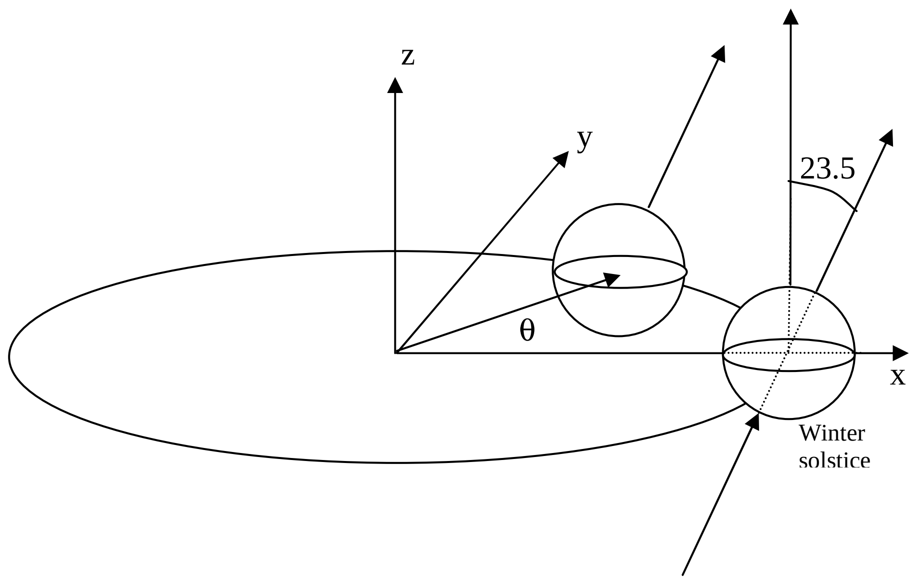
Here's a drawing that captures all that:



The earth shown at left is where it is at the winter solstice. You can see that the polar axis is tilted in the xz-plane. By the way, I've chosen to use coordinates centered at the sun, with the x-axis passing through the earth's center at the winter solstice, and the entire orbit lying in the xy-plane, which is (I believe) called "the plane of the ecliptic" by astronomers. The direction "z" is perpendicular to this plane.

After some number of days – about 55 in the drawing above – the earth has moved some part of the way around its orbit. Because its speed is assumed constant, that part is 55/365.25 of the way; in degrees, that would be 360 * 55/365.24 = 54.2 degrees. That angle is called "theta" in the picture, and that's how I'll be referring to it in general. But I'll write theta in *radians*, which are the mathematician's way of measuring angles. Instead of 360 degrees in the circle, there are 2pi (about 6.28) radian. So to find theta, I'd do the following:

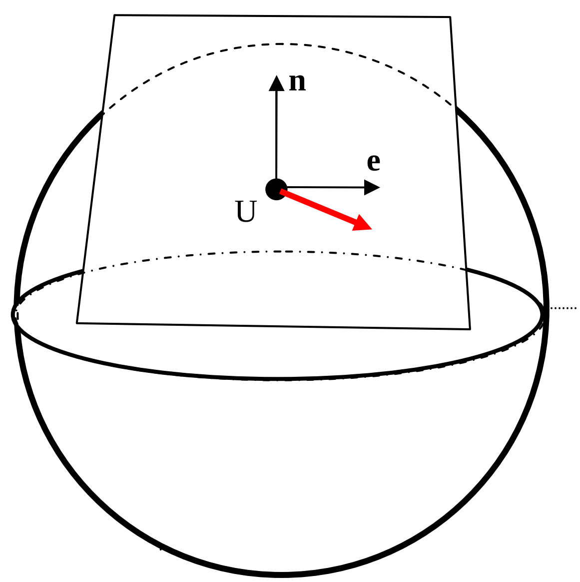
Theta = 2 * pi * (days since solstice) / 365.25.

I'm also going to give a name to the angle between the z-direction and the polar axis (i.e., 23.5 degrees). I'm going to call that alpha.

Next, I'm going to name our location as we observe the sunrise: we're at a point U on the earth's surface. I want to point out something important about U right here:

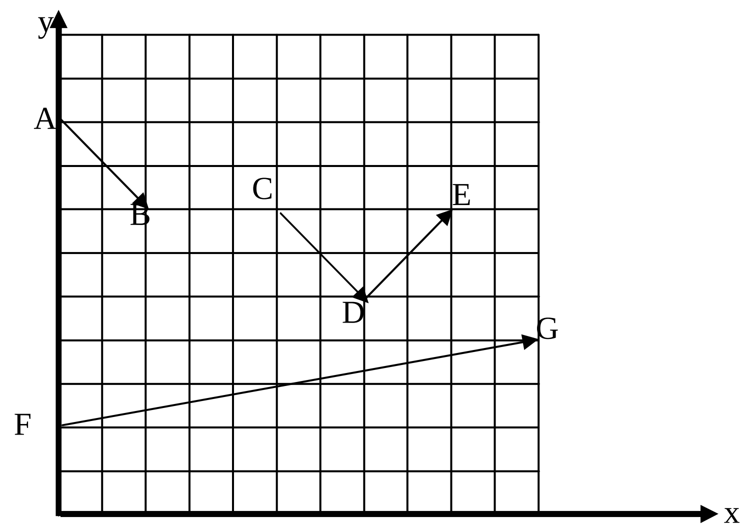
The ray from U to the center of the earth is perpendicular to the ray from U towards any direction on the horizon, and since it's sunrise, we know in particular that it's perpendicular to the ray from U to the sun. Again: the ray from U to the sun is orthogonal to the ray from U to the center of the earth.

I'm going to draw a picture of the earth, showing the point U and the plane that contains U but is perpendicular to the line from U to the earth's center (i.e., the "horizontal plane at U" or "tangent plane at U"):



I've also drawn, on that plane, a pair of arrows; the one labeled **n** points in the northerly direction from U (i.e., if you were standing at U and looked towards the horizon in the direction your compass called north, you'd be looking in the direction **u**); similarly the arrow labeled **e** points to the east in the tangent plane. The arrow from U to the sun *also* lies in this plane (I've drawn it in as a red arrow), and the big question we're trying to answer is "how many degrees is it from the ray labeled **n** around to the red ray, and is it clockwise or counterclockwise?"

With the definitions above made, so that we know what we're talking about in general, I need to digress briefly about vectors, since those are what I'll use to do the computations. (I'm a mathematician by training, and vector mathematics is infinitely easier for me that spherical trigonometry, so that's how I'll explain all this stuff.) Let me start in two dimensions. In the picture below, there's a grid and a few points, and a pair of axes labeled x and y. Let's look at how you get from one point to another:



From A to B requires that you go right two steps (or two steps in the "x direction" and go -2 steps in the y direction. So we call this displacement (2, -2). In the same way, to go from F to G is a displacement of (11, 2). What about C to D? It's (2, -2) – the same as the displacement from A to B. So although A and B are different points from C and D, the displacement from A to B is the same as the displacement from C to D. What about from D to E? That's (2, 2). What about C to E? That's just four steps to the right, i.e., (4, 0). Notice that if you take the displacement from C to D (2, -2) and the displacement from D to E (2, 2), and add up corresponding numbers, you get the displacement from C to E, namely (4,0).

These representations of displacements are called "vectors," and their written as pairs of numbers. The rule for adding vectors is that we add term-by-term, (as we did with C-to-D and D-to-E above) so that

$(a,b) + (c,d) = (a + c, b + d)$

We can also multiply a displacement: if we go from A to B (which is (2, -2)) and then go the same displacement twice more, we end up having traveled (6, -6). So we say that in general,

$n\,(a, b) = (na, nb)$

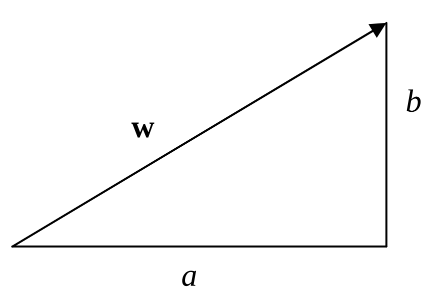
By the way, it's conventional to denote vectors with boldface letters, so that we might write **u** = (*a*, *b*), and then talk about the vector **u.** I'll also use italics to denote ordinary numbers, so that it's easy to tell the word "a" from a number denoted by *a*.

One more new idea needs talking about. You can see that it's possible for two vectors to be perpendicular (the vector from C to D and the vector from D to E, for instance). Is there a way to tell this by looking at the numbers that represent them? Yes, there is – and it works in 3 dimensions as well. Here's the deal:

If **u** = (*a, b*) and **v** = (*c, d*) are two vectors, then we'll say that the number *ac + bd* is their *inner product*. This number comes from multiplying corresponding terms (*a* and *c* are the first part of each vector, *b* and *d* are the second part) and then adding up the results. What does this inner product tell us about **u** and **v**? First of all, if it happens to be zero, then **u** and **v** are perpendicular. Let's see that in action: the vector from C to D is (2, -2), and the vector from D to E is (2, 2). The inner product is 2 x 2 + 2 x (-2) = 0. Try it with a few others to convince yourself.

If the inner product is zero, the vectors are perpendicular; if the vectors are perpendicular, then the inner product is zero.

Here's another application of inner products: how long is the vector **w** = (a.b)? Well, we can use Pythagoras's theorem to answer that. By drawing a triangle with **w** along the hypotenuse, and "a" along the bottom, and b as the right edge, we see that the length of w squared is a*a + b*b, which happen to be exactly <**w, w**>. That means that the length of **w** is just sqrt(<**w, w**>). It's usually written ||**w**||**.** A vector whose length is one is called a *unit vector*.



In general, if **u** and **v** are vectors, and the angle between them (when we move them so that they start at the same point) is *b*, then a wonderful theorem (which I won't try to prove here) shows that

(I) $<\mathbf{u}, \mathbf{v}> = ||\mathbf{u}||\ ||\mathbf{v}|| \cos(b)$

where "cos" is the cosine function on your calculator.

The inner product really is a little like multiplication. Recall that for numbers *x*, y, and *z*, we know that $x(y + z) = xy + xz$. Well, a similar thing is true for inner products:

(A) $<\mathbf{u}, \mathbf{v+w}> = <\mathbf{u,v}> + <\mathbf{u,w}>$
(B)

It's also true that if *a* is any number, then

(B) $<\mathbf{u}, a\mathbf{v}> = a <\mathbf{u,v}> = <a\mathbf{u}, \mathbf{v}>$

and that in general

(C) $<\mathbf{u.v}> = <\mathbf{v,u}>$

That's pretty much all we need to know about inner products.

That's just about everything you need to know about vectors…except that it all works in 3 dimensions as well. That is to say, we can treat displacements in 3D as being represented by displacements in *x*, *y*, and *z* (so you need three numbers to describe a vector), they still get added term-by-term, and they get scaled up or down by multiplying each term by the same number.

I forgot: there's one last thing you need to know. If you have a pair of non-parallel vectors, **f** and **g**, and you move them so that their starting points are the same, and you have a pair of numbers *s* and *t*, then the vector

$s\mathbf{f} + t\mathbf{g}$

actually lies in the plane that contains the vectors **f** and **g**.

With that in mind, we're ready to proceed.

Recall that the earth's center is at a point in the *xy*-plane, and that the sun is at the location (0,0,0). And the earth is at an angle θ around the circle from the *x*-axis. That means that a vector from the earth to the sun points in the direction

Sun-to-earth = $(\cos(\theta), \sin(\theta), 0)$.

You'll have to take that on faith, or plot some examples for a few values of theta to see that they all lie in a circle. Note that the vector above is a unit vector, for its length is cos(θ) * cos(θ) + sin(θ) * sin(θ) + 0*0, and the defining fact about sin and cos is that cos(t) * cos(t) + sin(t) * sin(t) is always one.

We'll be looking instead at a vector *from* the earth *towards* the sun, which is just the negative of the vector above:

s = unit vector pointing from earth towards sun = (-cos(θ), -sin(θ), 0)

Now let's look at the vector that points in the direction from the south pole to the north pole. I'm not going to write down the actual vector, which is 8000 miles long, but instead a vector of length 1 (i.e., a unit vector) that points in the same direction.

Look at the earth at the winter solstice. You can see that the "pole vector" lies in the *xz*-plane. In fact, it's tilted a bit from the straight-up-in-z vector. It turns out that it's exactly

p = (sin (α), 0, cos(α))

where (you'll recall) α is the angle between this polar vector and the straight-up vector. We'll use the vector **p** and the vector **s** a good deal. The other vector we'll use a lot is **u**, a unit vector that points from the center of the earth towards our location, *U*.

Now what can we do to discover a formula for the vectors **n** and **e**? Let's work on **n** for the moment. We know that **n** is in the plane that contains **u** and **p**. So we can write it down as a combination of **u** and **p**. Let's see if we can do it in the form

n = *c* **u + p.**

If we can find the right number c, we'll be in business. We'll soon see that we can't quite do so, but we can come pretty close.

What do we know about the vector **n**? It has to be perpendicular to **u**. That means that **<u,n>** = 0. That, in turn, means that

<**u,** *c* **u + p> = 0**

Now's the time that all those facts about inner products come into play:

0 = <**u,** *c* **u + p**> (from above)
= <**u,** *c***u> + <u,p> (**rule A)
= ***c*****<u,u**> + < **u,p**> (rule B)
= *c* + <**u,p**> (because **u** was chosen to be a unit vector, so <**u,u**> = 1)

Hence *c* = **-<u,p>**. Just what is this number? Well, recall (statement I) that

$<\mathbf{u},\mathbf{p}>. = ||\mathbf{u}||\ ||\mathbf{p}|| \cos(t)$

where t is the angle between **u** and **p**. But since **u** and **p** are both unit vectors, their lengths are both 1, so we find out that

$<\mathbf{u},\mathbf{p}>. = \cos(t)$

Now what is the angle between **u** and **p?** It's the complement of the angle between **u** and the equator, which is the latitude of the point U, i.e., *our* latitude. That means that $\cos(t) = \sin(h)$, where h is our latitude. So we can say that

$c = -\sin(h)$

To be more accurate, we've found that if we choose c to be $-\sin(h)$, then the vector

$c\,\mathbf{u} + \mathbf{p}$

will be perpendicular to **u**. But does it have the right length? Let's compute the *square* of its length, which is just its inner product with itself:

Length-squared
$= < c\,\mathbf{u} + \mathbf{p}, c\,\mathbf{u} + \mathbf{p}>$
$= c^2 <\mathbf{u},\mathbf{u}> + 2c <\mathbf{u},\mathbf{p}> + <\mathbf{p},\mathbf{p}>$
$= \sin(h)^2 - 2\sin(h) <\mathbf{u},\mathbf{p}> + 1$
$= \sin(h)^2 - 2\sin(h)^2 + 1$
$= 1 - \sin(h)^2$
$= \cos(h)^2$

In other words, its length is $\cos(h)$ instead of 1. By multiplying it by $1/\cos(h)$, we'll get a vector whose length is actually 1, and which is in the plane containing u and p, and which is perpendicular to u, which means it can only be one of two vectors (the "north" vector or the "south" vector. Because the amount of **p** in the vector is positive rather than negative, we'll know that we've got the north-pointing one. So here it is:

$\mathbf{n} = -\left(\sin(h) / \cos(h) \right) \mathbf{u} + (1 / \cos(h))\mathbf{p}$

At this point, we're a good part of the way to the solution. We know the vector **n**, and we know the vector **s**, so to find the angle β between **n** and **s**, i.e., the azimuth angle, we only need to compute their inner product:

$<\mathbf{n}.\mathbf{s}> = ||\mathbf{n}||\ ||\mathbf{s}|| \cos(\beta)$
$\quad = \cos(\beta)$ [because both are unit vectors]

So to find the angle β from **n** around to **s**, we just need to find the angle whose cosine is $<\mathbf{n},\mathbf{s}>$. Fortunately, that's what the arc-cosine function does.

So let's work out $\langle \mathbf{n},\mathbf{s} \rangle$. First, it's the same as $\langle \mathbf{s},\mathbf{n} \rangle$ because of property C. So

$$
\begin{aligned}
\langle \mathbf{n.s} \rangle = \langle \mathbf{s},\mathbf{n} \rangle
&= \langle s, -\left(\sin(h) / \cos(h) \right) \mathbf{u} + (1 / \cos(h))\mathbf{p} \\
&= \langle \mathbf{s}, -\left(\sin(h) / \cos(h) \right) \mathbf{u} \rangle + \langle \mathbf{s}, (1 / \cos(h))\mathbf{p} \rangle \\
&= -(\sin(h)/\cos(h)) \langle \mathbf{s},\mathbf{u} \rangle + (1/\cos(h)) \langle \mathbf{s},\mathbf{p} \rangle
\end{aligned}
$$

Now the first term involves $\langle \mathbf{s},\mathbf{u} \rangle$. But since **u** goes from the center of the earth to us, and **s** goes from us to the sun at sunrise, they are perpendicular, so this term is zero! That means that

$$\langle \mathbf{n.s} \rangle = (1/\cos(h)) \langle \mathbf{s},\mathbf{p} \rangle$$

Look at that. We know **s** – we figured it out at the very start up above: it was just $(-\cos(\theta), -\sin(\theta), 0)$. And we know **p** : it's $(\cos(\alpha), 0, \sin(\alpha))$. So their inner product is $-\cos(\alpha)\cos(\theta) + 0 + 0 = -\cos(\alpha)\cos(\theta)$. And we know *h* because it's our latitude. So we've got everything we need:

$$\cos(\beta) = -\cos(\alpha)\cos(\theta)/\cos(h)$$

hence

$$\beta = \operatorname{acos}(-\cos(\alpha)\cos(\theta)/\cos(h))$$

Now the only thing remaining is to tell whether the angle looks like N 24 E, or N 24 W. Fortunately, in this case that's very easy: the sun rises in the east and sets in the west. So the morning azimuth will be N xx E, and the evening one will be N xx W. For more complex situations, you need to use slightly more complex methods. Anyhow, our situation is a simple one: if the spreadsheet says "sunrise", we use "E", and similarly for "sunset" and "W".

And that's it – the whole technique in a (large) nutshell.