Analysis of a Mask-Based Nanowire Decoder*

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Abstract

A key challenge facing nanotechnologies will be controlling nanoarrays, two orthogonal sets of nanowires that form a crossbar, using a moderate number of mesoscale wires. Three methods have been proposed to use mesoscale wires to control individual nanowires. The first is based on nanowire differentiation during manufacture, the second makes random doped connections between nanowires and mesoscale wires, and the third, a mask-based approach, interposes high-K dielectric regions between nanowires and mesoscale wires. All three addressing schemes involve a stochastic step in their implementation. In this paper we analyze the mask-based approach and show that a large number of mesoscale control wires is necessary for its realization.

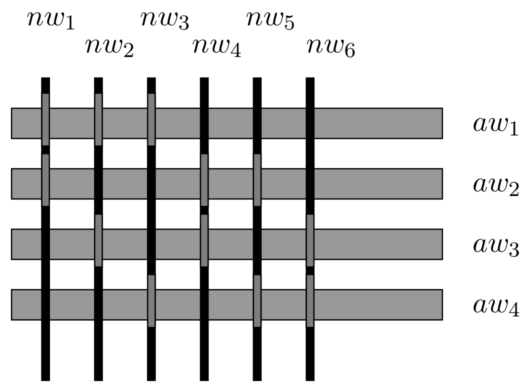

Figure 1. A decoder to address six (vertical) NWs with four (horizontal) mesoscale wires (MWs). The (highlighted) lightly doped regions of a NW allow activation of any two MWs to activate a unique NW [6].

1 Introduction

The crossbar, a simple but well-known connection network, consists of two sets of orthogonal wires. Switches are positioned at the crosspoints defined by the intersection of pairs of wires. Crossbars can be used as switching networks, memories, and programmed logic arrays.

Chemists have developed methods to assemble nanowires (NWs) into crossbars [2, 17, 19, 22]. They have realized switches by placing a thin layer of bistable molecules, such as rotaxanes or [2]-catenanes, between two orthogonal sets of NWs [3, 4]. When a large positive or negative electric field is applied between the NWs forming a crosspoint, the molecules at the crosspoint become either conducting or nonconducting. Once the state of these molecules is set, a smaller electric field can be used to sense their state without changing it by observing the level of current flowing through the crosspoint.

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Several methods have been devised to produce NWs using laser ablation [18], chemical vapor deposition (CVD) [5], superlattice NW pattern transfer (SNAP) [17], and nanolithography [16, 19]. Other methods for producing NWs are likely to be developed.

Controlling NWs with mesoscale wires (MWs) without losing the high crosspoint density afforded by small NW sizes offers an important challenge that can be met by a) positioning MWs at right angles to the NWs and b) using MWs to apply electric fields to lightly doped regions of NWs. The application of an electric field by a MW to an exposed and doped NW drives the conductance of that NW low. (See Fig. 1.) That is, NWs combined with MWs form Field Effect Transistors (FETs). A circuit that causes one NW to be conductive while the others are non-conductive by associating each NW with a unique set of active MWs is called a **decoder**.

As explained in Section 2, three methods have been proposed to control NWs with MWs. The number, M, of MWs needed to control N NWs with probability at least $1 - \epsilon$ has been determined analytically for the first two methods [7, 9, 21]. Here we analyze the third. For very reasonable assumptions M must be at least $2 \log_2 N + 46$. As NW dimensions decrease, M could easily grow to $2 \log_2 N + 300$.

The **mask-based decoder**, which is explained in detail in Section 2, is designed [12, 13] to work with undifferentiated NWs of the kind produced in the SNAP process [17]. It uses lithographically defined rectangular regions of high-K dielectric to shield NWs from the fields associated with MWs. Because lithography puts a lower limit on the size of such regions, the smallest regions are shifted randomly to differentiate NWs with respect to MWs. While this makes it possible to control NWs with MWs, as indicated above, a price must be paid in the high number of MWs needed for this task.

In Section 3 we explore the conditions that must be met for NWs to be independently addressable. We observe that these conditions are equivalent to solving variants of the coupon collector problem. In Section 4 we examine three models for the random shifting of dielectric regions. These models capture the essential details of methods proposed for laying down the smallest dielectric regions. Once regions have been imposed on NWs, it is possible that not all NWs will be controllable. In Section 5 we analyze the masked-based decoder on our three models. Conclusions are given in Section 6.

2 Three Methods of Addressing Nanowires

Lieber *et al* have shown that NWs can be assembled into crossbars using fluidic methods [15, 20]. When manufactured, NWs are doped in sections along their length [14], a process known as "modulation doping." A small number of NWs is drawn from a large ensemble at random and arranged in parallel on a chip. The NWs are addressed with MWs, as in Fig. 1. This **encoded NW decoder** is analyzed in [7, 9]. Dehon *et al* [7] show that N NWs can be uniquely addressed more than 99% of the time using M MWs where $M \geq \lceil 2.2 \log_2 N \rceil + 11$ MWs. A substantial amount of circuitry is required to map external addresses to internal ones. Gojman *et al* [9] examine the area required for such circuitry for a variety of decoding strategies.

The NW differentiation-based decoder requires that NWs be grown with doped regions. Other decoding strategies have been proposed that do not require that NWs be differentiated at the time of their manufacture. Williams and Kuekes [21] have proposed a **randomized contact decoder** in which N NWs are addressed by M MWs by making random doped contacts between MWs and NWs with probability of about 1/2; if a contact is made, it is assumed that the NW is controlled by the MW by one of some variety of means including FETs. The authors state that with probability at least .99, $M \geq 5 \log_2 N$ MWs suffice to provide unique addresses to all N NWs.

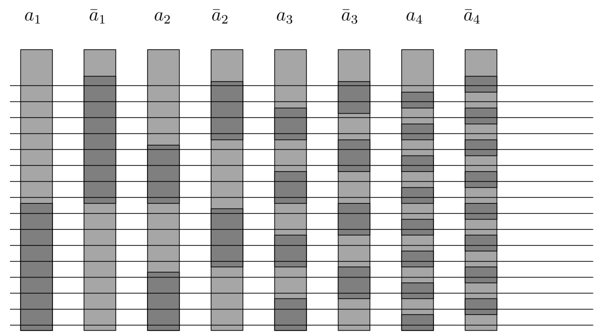

Figure 2. A decoder in which vertical MWs apply fields to horizontal NWs that are shield by dielectric regions (dark rectangles). Each NW exposed to a field becomes non-conducting. For each NW there are four MWs which, if activated, leave that wire conducting but cause all other NWs to be nonconducting.

The third decoder [12], called a **mask-based decoder**, places high-K dielectric regions defined lithographically (we call them LRs) in between MWs and lightly doped NWs, as suggested in Fig. 2. Such a region shields a NW from a field applied by a MW. An ideal mask-based decoder for $N = 2^k$ NWs has k pairs of LRs. For $0 \leq j \leq \log_2 N - 1$ the two sets in the jth pair each have 2^{j-1} LRs that cover complementary halves of the NWs (See Fig. 2). If a field is applied to one of the two MWs associated with a pair of LRs, exactly half of the NWs remain conducting. Because the LRs in a pair identify half of the NWs identified by the previous pair, an ideal mask-based decoder can select exactly one NW to remain conducting when a field is applied to one MW in each of the k pairs of MWs. Thus, an ideal mask-based decoder assigns a virtual address to each of the N undifferentiated NWs using $2 \log_2 N$ MWs.

A problem with this ideal mask-based decoder is that LRs whose width is the pitch of NWs cannot be lithographically produced; there is a lower limit, w, on the width and separation of LRs measured as multiples of the pitch of NWs. The mask-based decoder uses sets of LRs of width and separation w in which one LR is offset from the other by a distance which is either one NW pitch or a small multiple of a NW pitch. (See Fig. 3.) A set of LRs each offset from the other by some fixed distance is called a **cycle**.

The mask-based decoder assumes that one or more cycles will be placed on a mask and that one or more copies of a mask will be used. Taking into account the imprecision of lithography, we assume that the exact locations of the left edge of all LRs in a mask will be random variables and that the exact location of a mask relative to NWs on a chip will

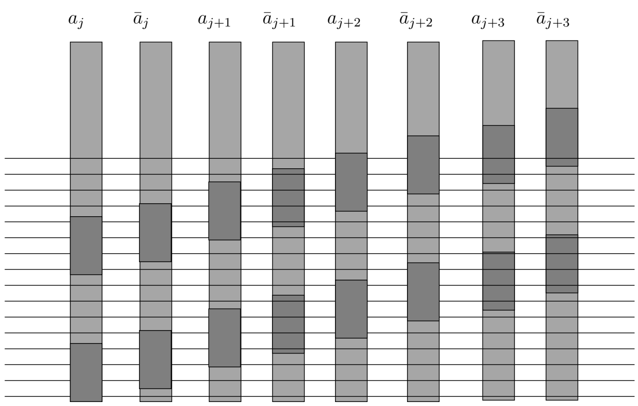

Figure 3. Smallest dielectric regions uniformly shifted about their evenly-spaced nominal positions.

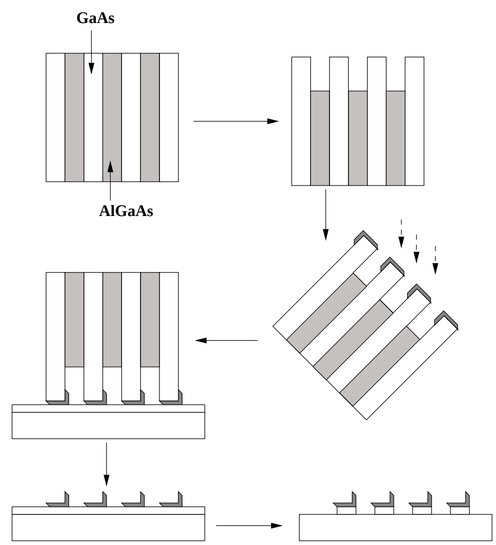

Figure 4. Diagram showing the SNAP process: The GaAs/AlGaAs superlattice; the superlattice after etching the AlGaAs layers; metal deposition at $36°$; transfer of metal onto adhesive layer on silicon; release of metal wires; removal of excess adhesive.

also be a random variable. In fact, we assume that the displacement of a mask will be a uniform random variable over one or more NW pitches.

2.1 The SNAP Process

Although the mask-based decoder has been proposed to control the long, straight and uniformly-spaced, lightly doped NWs produced by the superlattice nanowire pattern transfer (SNAP) method [17], it can be used with NWs produced by other methods.

SNAP uses molecular beam epitaxy to make a GaAs/AlGaAs superlattice from which the AlGaAs layer is etched back, filled with a metal through evaporation, and then pressed onto an adhesive layer on silicon. (See Fig. 4.) After the superlattice is removed, metallic NWs remain attached to the silicon [11]. The metallic NWs can be used as a nanometer-scale mask for a thin silicon layer residing on top of silicon oxide to produce silicon NWs. Although this work is experimental, the authors have shown that on the order of 100 or less of very long (2-3 mm), small (8-10 nm), and largely defect-free NWs having a uniform pitch (16-60 nm) can be deposited on a chip with each application of SNAP.

In E-beam lithography, which is being used to test the concept of the mask-based decoder, the length of and spacing between LRs are known to within about 10 nms whereas the offset of a mask producing a set of LRs can vary by 50 to 100 nms [1]. In these tests the NW width and separation are each about 15nms. E-beam lithography is currently too expensive for mass production. If photolithography is used, the uncertainty in the length and separation of LRs can be four or five times as great.

It is possible that the LRs in a mask-based decoder would be placed using a stamping process [1]. We believe that errors introduced by this process can be modeled and analyzed using our methods.

3 Controllability of NWs

To simplify the discussion of a mask-based decoder, a region of a NW is said to be **doped** if it is fully exposed to the electric field of one of the MWs that lies across it and **undoped** if it is fully covered by a LR. If an LR only partially covers a NW, it and the corresponding MW are said to have **failed** because the corresponding MW has a partial effect on the NW in question when it should either have no effect or completely control the NW. We assume that we can detect those LRs that partially cover a NW so that they can be left unused.

A NW is conducting only when its doped regions are adjacent to MWs that do not carry electric fields. Let A and B be two NWs and let D_A and D_B be the set of doped regions of NWs A and B corresponding to the M MWs. We show that if $D_A \subseteq D_B$, then NWs A and B cannot be controlled independently. If $D_A = D_B$, the result is obvious. If $D_A \subset D_B$ and the MW fields are such that NW B is conducting (no fields are applied to its doped regions), then NW A is also conducting. However, NW A can be conducting when NW B is non-conducting. Since this leads to an ambiguous result when reading and writing data, we say that NW B is not controllable if there is a NW A such that $D_A \subseteq D_B$.

Lemma 3.1 *All NWs are independently controllable if for no two NWs A and B is $D_A \subseteq D_B$. Equivalently, all NWs are controllable if each is the rightmost and leftmost NW in two different LRs.*

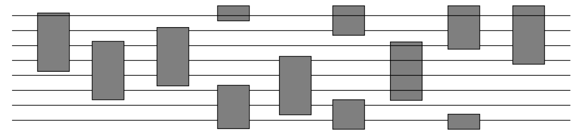

Figure 5. Repeated dielectric regions modeled as cyclic patterns.

4 Models for Region Displacement

We consider three models for the location and displacement of the LRs in a mask-based decoder when the smallest LRs have width and separation of w NW pitches. We assume that each model uses $2\log_2(N/2w)$ MWs to identify a set of $2w$ NWs that remain conducting. To resolve the control down to the level of an individual NW, an additional T MWs are used for a total of $M = 2\log_2(N/2w) + T$ MWs. The supplemental number of MWs, T, depends on the assumptions made concerning the uncertainty in the offsets of masks from NWs and the offsets of LRs within masks.

There is also uncertainty regarding the locations of boundaries of LRs associated with the first set of $2\log_2(N/2w)$ MWs used to identify sets of $2w$ NWs. Fortunately, this uncertainty can be dealt with using only the supplemental set of T MWs. At the LR boundaries associated with the first set of MWs a small number of NWs, say q, may be always on or off or fail. To compensate for this uncertainty, the supplemental MWs can be used to always turn off the q NWs at each boundary of a set of $2w$ NWs, thereby reducing the controllable number to $2(w-q)$ in each group of $2w$ NWs. Thus, instead of controlling N NWs, $N(w-q)/w$ are controlled. One can compensate for this by increasing N by a factor of $w/(w-q)$.

To resolve a set of $2w$ NWs down to one NW, each model assumes that a) m masks are used, b) each mask contains n LR cycles where a cycle has r LRs nominally shifted one from the other by a distance of $2w/r$ NW pitches (r divides w), and c) that there is uncertainty in the positions of LRs within masks and of masks relative to NWs. In the first model and third models, $r = 2w$. In the second, $r = 1$.

Each of the three models assumes that the offsets of the masks from NWs are statistically independent and identically distributed (i.i.d.) random variables with a uniform distribution over either one NW pitch or multiple NW pitches. In both cases we let $-1/2 \leq \theta \leq 1/2$ denote the fractional displacement of a mask with respect to the range of the random variable where $\theta = 0$ corresponds to nominal position of the left boundary of an LR being in the center of a NW. Since θ has a uniform distribution, the probability that the nominal position of the left boundary of a LR falls inside a NW (it fails) is 1/2.

Each model also assumes that the offsets of the left boundaries of LRs relative to their nominal locations on a mask are i.i.d. random variables with continuous distribution $p_{off}(x)$. The value of $p_{off}(x)$ depends on the model.

Below we compute the probability that all NWs can be controlled even when it is possible that some MWs fail. We determine conditions under which this probability approaches 1 as the number of MWs (and LRs) increase.

When there is no variation about the nominal position of each LR and their width and spacing are the same, all the MWs will fail with probability 1/2. This probability could be reduced to zero if a mask contains two sets of uniformly spaced LRs, one offset from the other by half a NW pitch. We assume that this is not a realistic possibility when photolithography is used and the NW pitch is under 15-20 nm.

First Model: Our first model is consistent with the first experimental verification of the mask-based decoder [1] in which the target width and pitch are about 15nm and 30nm, respectively, and the uncertainty in an LR offset within a mask is about 10 nm. It assumes that the NWs are large enough relative to the width and separation that a) LRs in a cycle can be placed to within one NW pitch ($r = 2w$), b) $p_{off}(x)$ is uniformly distributed over the interval $[-\rho/4, \rho/4]$ where ρ is the NW pitch, and c) that the offset of a mask follows a uniform distribution over one NW pitch centered on the middle of a NW. If the global offset of a mask places the left boundaries of LRs exactly in the middle of NWs, all MWs fail. However, this will occur with an infinitesimal probability.

Second Model: The second model is used primarily to introduce a key method of probabilistic analysis. It assumes that a) each mask has one LR ($r = 1$) and b) the offset of a mask is equally likely to be anywhere in $2w$ NW pitches. It is analyzed using the cyclic LR model depicted in Fig. 5. The probability that a given MW fails (the left boundary of its corresponding LR falls on a NW) is one half.

Third Model: The third model is the same as the first except that a) we assume that each mask has one cycle ($n = 1$), b) the offset of a mask relative to NWs is equally likely to be anywhere within a NW pitch, and c) the uncertainty in the location of LRs relative to nominal locations on a mask is large and follows a potentially non-uniform probability distribution $p_{off}(x)$ whose support (interval over which it is non-zero) can be more than one NW pitch. We expect the effect of mask offset, denoted by θ, on the probability that an LR boundary moves from its current location between two NWs to the jth space to the right or left is small. We let this probability be fixed at p_j where $\sum_j p_j = 1/2$.

4.1 Variable-Length Lithographic Regions

In Lemma 3.1 we show that NWs are independently controllable if each is both a rightmost and a leftmost NW in an LR. When LRs have fixed length which is a multiple of a NW pitch, an LR's left endpoint falls between two NWs if and only its right endpoint does as well. Thus, for all NWs to controllable, it is sufficient to consider only the left endpoints of LRs as modeled by the coupon collector problem. If LRs are of variable length, the left and right endpoints must be considered separately. In the worst case, they would behave completely independently and the number of trials to collect all coupons would be doubled.

5 Analysis of the Displacement Models

The goal in each of the displacement models is to determine T, the supplemental number of MWs required for all N NWs to be controllable. As shown in Lemma 3.1 this is equivalent to ensuring that each pair of adjacent NWs has some LR's left endpoint between them. This can be modeled by variants of the standard **coupon collector problem**, a classical problem described in [8]. The standard problem is to determine the number of trials T needed to collect C coupons with probability at least $1-\epsilon$ when each coupon is equally likely to be drawn on each trial. Here a coupon is collected when an LR's left boundary falls between a specific pair of adjacent NWs. There is one coupon for each pair of adjacent NWs. In each model additional coupons are introduced to model each LR failure. It is straightforward to show that T must grow as $C \log C$. Variants on the coupon collector problem also arise in the analysis of the encoded NW decoder [9].

5.1 Analysis of the First Displacement Model

The first displacement model assumes that n cycles of $2w$ LRs are placed on each of m masks. Offsets of LR left boundaries from their nominal locations within masks are i.i.d. random variables with distribution $p_{off}(x)$ whose support is a fraction of a NW pitch. Offsets of masks from NWs are also uniform i.i.d. random variables distributed over a NW pitch. This model uses $T_1 = 2w(nm)$ MWs.

Let $P_{first}(T_1, C, n, m)$ be the probability that not all $2w$ NWs are controllable. We find n and m such that this probability is largest when nm, the number of cycles, is fixed.

Lemma 3.1 shows that all $2w$ NWs in a cycle can be controlled if and only if the left boundary of some LR falls between every pair of NWs. When $p_{off}(x)$ is 0 outside of the interval $[-\frac{\rho}{4}, \frac{\rho}{4}]$, only one LR in each cycle has a left endpoint that can fall between any two adjacent NWs. If there are n cycles on a mask, there are n such LRs. If there are m masks with n cycles per mask, there are mn LRs that can fall between any two adjacent NWs. The n candidate LRs on one mask experience the same mask offset. Those on different masks may experience different offsets.

Let θ denote the offset of the nominal location of an LR in a mask relative to a NW and let $p_{fail}(\theta)$ denote the conditional probability that it fails by having its left boundary LR overlap a NW. $p_{fail}(\theta) = \int_{\theta-1/4}^{\theta+1/4} p_{off}(x)dx$ where $p_{off} = 0$ outside of the interval $[-\rho/4, \rho/4]$.

If $p_{off}(x)$is uniform over $[-\frac{\rho}{4}, \frac{\rho}{4}]$, $p_{off}(x) = 2/\rho$. Thus, $p_{fail}(0) = 1$ (no displacement of the left boundary of an LR relative to its nominal causes it to leave a NW), at $\theta = \pm 1/2$, $p_{fail}(\theta) = 0$ (the left boundary of an LR cannot leave the space between NWs), and in between $p_{fail}(\theta)$ is monotonically increasing. More generally, $p_{fail}(\theta) = |2\theta - 1|$.

Given adjacent NWs, there are n LRs on one mask whose left boundaries nominally fall between their midpoints. Let $q(\theta)$ be the conditional probability that these LRs fail to fall in the space between the NWs when the mask offset is θ. Then, $q(\theta) = p_{fail}^n(\theta)$. Let $q(\theta_1, \ldots, \theta_m)$ be the conditional probability, given offsets $\theta_1, \theta_2, \ldots, \theta_m$ of the m masks, that there are nm LRs in all m masks that fail to fall in the same space. Then, $q(\theta_1, \ldots, \theta_m) = p_{fail}^n(\theta_1) \cdots p_{fail}^n(\theta_m)$.

All $2w$ NWs cannot be controlled unless the left boundary of some non-failing LR falls between each pair of adjacent NWs. Conditioned on $\theta_1, \ldots, \theta_m$, LRs fail independently. Hence the probability all $2w$ NWs are controllable, given $\theta_1, \ldots \theta_n$, is $(1 - q(\theta_1, \ldots, \theta_m))^{2w}$. Averaging this probability over all (uniform) θ_i gives $1 - P_{first}(T, C, n, m)$.

Applying $1 - sx \leq (1-x)^s \leq 1 - sx + \binom{s}{2}x^2$ for $0 \leq x \leq 1$ to $(1 - p_{fail}(\theta_1)^n \cdots p_{fail}(\theta_m)^n)^{2w}$ and using $\binom{s}{2} \leq s^2/2$, we have the following bounds where $X = (2w)\left(\int_0^1 p_{fail}^n(\theta)d\theta\right)^m$, $Y = (2w^2)\left(\int_0^1 p_{fail}^{2n}(\theta)d\theta\right)^m$, and $s = 2w$.

$$X - Y \leq P_{first}(T_1, C, n, m) \leq X$$

Lemma 5.1 *When* $Y \leq X/2$, $P_{first}(T_1, C, n, m)$ *is at least* $X/2$ *and at most* X *where* $X = (2w)\left(\int_0^1 p_{fail}^n(\theta)d\theta\right)^m$.

The lemma is illustrated by the example of $p_{off}(x)$ that is uniform over $[-\frac{\rho}{4}, \frac{\rho}{4}]$ in which case $p_{fail}(\theta) = |2\theta - 1|$. Through a straightforward change of variables, it is easy to see that $X = 2w(1/(n+1))^m$ and $Y = 2w^2(1/(2n+1))^m$. Thus, $Y = w((n+1)/(2n+1))^m X \leq w(2/3)^m X$ since

$n \geq 1$. [$(n+1)/(2n+1)$ approaches 1/2 as n increases.] Thus, if $m \geq \ln(2w)/\ln(3/2)$, $Y \leq X/2$. In this case, $X/2 \leq P_{first}(T_1, C, n, m) \leq X$.

Under the conditions of the lemma, we are in a position to discuss the values of n and m for which $P_{first}(T_1, C, n, m)$ is smallest under the assumption that the number of cycles used in all masks, nm, is fixed. We need only consider the effect of values of n and m on $X/(2w) = \left(\int_0^1 p^n_{fail}(\theta)d\theta\right)^m$.

Chebyshev's inequality [10] states that $\left(\int_0^1 p_{fail}(\theta)d\theta\right)^n \leq \int_0^1 p^n_{fail}(\theta)d\theta$ for all integers $n \geq 1$. It follows that $X/(2w) \geq \left(\int_0^1 p_{fail}(\theta)d\theta\right)^{mn}$ which means that placing cycles on separate masks decreases $P_{first}(T, C, n, m)$, at least to the extent that $P_{first}(T, C, n, m)$ is well approximated by X. We summarize this result below.

Theorem 5.1 *Let* $(2w)^{1/m}\int_0^1 p^{2n}_{fail}(\theta)d\theta \leq \int_0^1 p^n_{fail}(\theta)d\theta$. *Then* $P_{first}(T_1, C, n, m)$, *the probability that not all NWs can be controlled in the first model, satisfies* $X/2 \leq P_{first}(T_1, C, n, m) \leq X$ *where* $X = 2w\left(\int_0^1 p^n_{fail}(\theta)d\theta\right)^m$.

When the number of cycles used in the model, $c = mn$, *is fixed, both bounds are maximized under variation of* n *and* m *when* $n = 1$. *Under this condition,* T_1, *the number of MWs required to assure that not all NWs can be controlled with probability* $P_{first}(T_1, C, n, m)$ *satisfies the following bounds where* $Z_1 = (P_{first}(T_1, C, 1, m))^{-1}$.

$$(2w)\log_2(wZ_1) \leq T_1 \leq (2w)\log_2(2wZ_1)$$

The total number of MWs required under these conditions is $M = 2\log_2 N/(2w) + T_1$.

Proof Under the conditions of the theorem, $X/2 \leq P_{first}(T_1, C, n, m) \leq X$. Under the assumption that NW width and separation are the same, $\int_0^1 p_{fail}(\theta)d\theta = 1/2$, from which the conclusion follows. ■

Thus, we arrive at a surprising and counterintuitive result, namely, the probability $P_{first}(T_1, C, n, m)$ is minimized when mn is fixed by setting $n = 1$, that is, by placing only one cycle on a mask thereby increasing the uncertainty in the locations of LRs.

When $p_{off}(x)$ is uniform over the interval $[\frac{-\rho}{4}, \frac{\rho}{4}]$ and $P_{first}(T_1, C, 1, m) = .01$ and $w = 3$, the number of MWs M satisfies $2\log_2 N + 46 \leq M \leq 2\log_2 N + 53$. When $w = 5$, $2\log_2 N + 86 \leq M \leq 2\log_2 N + 96$.

5.2 Analysis of the Second Displacement Model

The performance of the second region displacement model is captured by the **partial standard coupon collector problem**, a variant of the coupon collector problem. We associate one bin with each of the $2w$ NWs and each of the $2w$ spaces between them. Thus, there are $C = 4w$ bins, or coupons. On each trial, each bin is equally likely to be filled with a ball. We associate the first $C/2$ bins with spaces between NWs and the second $C/2$ bins with the NWs themselves.

In accordance with Lemma 3.1, the goal in the second displacement model is to have enough LRs so that the left boundary of some LR falls between every two adjacent NWs. In the standard coupon collector problem, the goal is to fill all C bins (collect all C coupons). Here the goal is to determine the number of trials required to fill the first $C/2$ bins with probability at least $1 - \epsilon$.

Theorem 5.2 *The number of trials,* T_2, *necessary to ensure that the first* $C/2$ *coupons are collected with probability* $1 - P_{partial}(T_2, C)$ *in the partial standard coupon collector problem satisfies the following bounds when* $C \geq 2$ *and* $P_{partial}(T_2, C) \leq 1/3$ *where* $Z_2 = (2P_{partial}(T_2, C))^{-1}$.

$$\frac{C}{(1+1/C)}\ln\frac{CZ_2}{(1+P_{partial}(T_2, C))} \leq T_2 \leq C\ln(CZ_2)$$

The total number of MWs is $M = 2\log_2 N/(2w) + T_2$ *where* $C = 2w$.

Proof Let E_k be the event that a ball does not fall in bin k in T_2 trials when there are C bins. It follows that $P_{partial}(T_2, C) = P(E_0 \cup E_1 \cup \ldots \cup E_{C/2-1})$ The principle of inclusion and exclusion, stated below, is used to derive bounds on this probability. (The sum on $k < l$ has $\binom{C/2}{2}$ terms.) $\sum_{k=0}^{C/2-1} P(E_k) - \sum_{k<l} P(E_k \cap E_l) \leq P_{partial}(T_2, C) \leq \sum_{k=0}^{C/2-1} P(E_k)$ Here $P(A)$ is the probability of event A.

For event E_k to occur, a ball (the left boundary of a LR) must fall in some bin other than bin k on each of the T_2 trials. Because the selection of each bin is statistically independent of all other selections and each outcome is equally likely, $P(E_k) = (1-1/C)^{T_2}$. For events E_k and E_l to jointly occur on each of the T_2 trials, a ball must fall in a bin other than the kth and lth. By the same reasoning it follows that $P(E_k \cap E_l) = (1-2/C)^{T_2}$. Using the additional facts that $(1-2/C) \leq (1-1/C)^2$ and $\binom{C/2}{2} \leq (C/2)^2/2$ we have $z(1-z/2) \leq P_{partial}(T_2, C) \leq z$ where $z = (C/2)(1-1/C)^{T_2}$. To solve for C, let $P_{partial}(T_2, C) = \epsilon$. The lower bound implies that $z \leq (1-\sqrt{1-2\epsilon})$. Since this bound on z is at most $\epsilon(1+\epsilon)$ when $\epsilon \leq 1/3$, we have that $z = (C/2)(1-1/C)^{T_2}$ satisfies $\epsilon \leq z \leq \epsilon(1+\epsilon)$. Finally, because $-x(1+x) \leq \ln(1-x) \leq -x$ when $x = 1/C \leq .5$ or $C \geq 2$, we have that $(C/2)e^{-(T_2/C)(1+1/C)} \leq z \leq (C/2)e^{-(T_2/C))}$ from which the desired conclusion follows. ■

When $P_{partial}(T_2, C) = .01$, $w = 5$ the number of

MWs M satisfies $2\log_2 N + 120 \le M \le 2\log_2 N + 135$ and when $w = 10$, $2\log_2 N + 291 \le M \le 2\log_2 N + 299$.

5.3 Analysis of the Third Displacement Model

The third displacement model, which assumes that the left boundary of a next LR is nominally shifted to the right from the previous one by a NW pitch, is captured by the **cyclic coupon collector problem**. We assume that m sets of $2w$ LRs located in one mask are deposited on a chip. Using the same notation as the previous model, the cyclic coupon collector problem aims balls at the first $C/2$ bins in succession, repeating this process m times for a total of $T_3 = m(C/2)$ trials. Although a ball is aimed at one of the first $C/2$ bins, it may also fall into a different bin, including one of the last $C/2$ bins that corresponds to a failure.

We assume that a ball aimed at bin i falls in bin $(i + j) \bmod (C/2)$, $0 \le j \le C/2 - 1$ with probability p_j. The probability that a ball doesn't fall in bin k on trial r, $0 \le r \le m(C/2) - 1$, is $(1 - p_{i(r,k)})$ where $i(r,k)$ satisfies $(r + i(r,k)) \bmod (C/2) = k$. We derive bounds on $P_{cyclic}(T_3, C)$, the probability that not all of the first $C/2$ coupons are collected in the cyclic coupon collector problem in $T_3 = m(C/2)$ trials. Thus, $P_{cyclic}(T_3, C) = P(E_0, E_1, \ldots, E_{C/2-1})$.

Observe that if $p_j = 1/C$ for all j, this problem is identical with the partial standard coupon collector problem.

As r ranges from 0 to $m(C/2) - 1$, $i(r,k)$ covers all values in $\{0, 1, \ldots, C/2 - 1\}$ m times. Because the trials are i.i.d., $P(E_k) = \left(\prod_{r=0}^{C/2-1}(1 - p_{i(r,k)})\right)^m = \left(\prod_{u=0}^{C/2-1}(1 - p_u)\right)^m$, which is independent of k.

Since we use the principle of inclusion and exclusion to bound $P_{cyclic}(T_3, C)$, we also need to compute the probability $P(E_k \cap E_l)$ that bins k and l do not contain balls at the end of T_3 trials.

On the rth trial balls fall into bins other than k and l with probability $(1 - p_{i(r,k)} - p_{i(r,l)})$ where $i(r,k)$ and $i(r,l)$ satisfy $(r + i(r,k)) \bmod (C/2) = k$ and $(r + i(r,l)) \bmod (C/2) = l$. Observe that $i(r,k)$ and $i(r,l)$ are cyclic, that is, $i(r,k)$ and $i(r,l)$ range over all values in the set $\{0, 1, \ldots, C/2 - 1\}$ as r ranges from 0 to $C/2 - 1$. Because the trials are statistically independent, $P(E_k \cap E_l) = \left(\prod_{r=0}^{C/2-1}(1 - p_{i(r,k)} - p_{i(r,l)})\right)^m$.

We use the inequality $(1 - p - q) \le (1 - p)(1 - q)$, $0 \le p, q \le 1$, to overbound $P(E_k \cap E_l)$ and underbound $P_{cyclic}(T_3, C)$. Because $\{p_{i(r,k)} \mid 0 \le r \le C - 1\} = \{p_0, p_2, \ldots, p_{C/2-1}\}$, the upper bound can be further simplified to $P(E_k \cap E_l) \le \left(\prod_{r=0}^{C/2-1}(1 - p_u)\right)^{2m}$, which is the square of $P(E_k)$.

The displacement probabilities $\{p_0, p_1, \ldots, p_{C/2-1}\}$ determine the accuracy with which the edge boundaries fall between particular NWs.

Theorem 5.3 *Let $\gamma = -\sum_{j=0}^{C/2-1} \ln(1 - p_j)$. The number of trials, T_3, necessary to ensure that the first $C/2$ coupons are collected with probability $1 - P_{cyclic}(T_3, C) \ge 2/3$ in the cyclic coupon collector problem satisfies the following bounds where $\sum_i p_i = 1/2$ and $Z_3 = (2P_{cyclic}(T_3, C))^{-1}$*

$$\frac{C}{\gamma} \ln\left(\frac{CZ_3}{(1 + P_{cyclic}(T_3, C))}\right) \le T_3 \le \frac{C}{\gamma} \ln(CZ_3)$$

Here $1 + \sigma/2 \le \gamma \le 1 + \sigma$ where $\sigma = \sum_{i=0}^{C-1} p_i^2 \le .25$. The total number of MWs required under these conditions is $M = 2\log_2 N/(2w) + T_3$.

Proof To bound $P_{cyclic}(T_3, C)$, we apply the principle of inclusion/exclusion described above. Given the expressions derived for $P(E_k)$ and $P(E_k \cap E_l)$ the following bounds apply where $z = (C/2)e^{-\gamma m}$ and $m = T_3/C$.

$$z(1 - z/2) \le P_{cyclic}(T_3, C) \le z$$

Following the proof of Theorem 5.2, we have that z satisfies $P_{cyclic}(T_3, C) \le z \le P_{cyclic}(T_3, C)(1 + P_{cyclic}(T_3, C))$ when $P_{cyclic}(T_3, C) \le 1/3$. The bounds on T_3 follow from these observations.

Bounds on the function γ can be derived using the inequality $\ln(1 - x) \le -x(1 + x/2)$, which holds for all values of x, and $\ln(1 - x) \ge -x(1 + x)$, which holds for $0 \le x \le .65$. Thus, it follows that γ satisfies $1 + \sigma/2 \le \gamma \le 1 + \sigma$ because $p_j \le .5$ where $\sigma = \sum_{i=0}^{C-1} p_i^2$. Finally, σ is largest under the constraint $\sum_j p_j = .5$ when all the mass of the distribution is placed on one value of j, that is, $\sigma = .25$. ■

Since $1 \le \gamma \le .125$, when $P_{cyclic}(T_3, C) = .01$ and $w = 5$ the number of MWs is between $2\log_2 N + 109$ and $2\log_2 N + 137$. When $w = 10$, it is between $2\log_2 N + 238$ and $2\log_2 N + 299$.

It is of interest to know how sensitive the bounds on T_3 are to the probability distribution $\{p_0, p_1, \ldots, p_{C-1}\}$. When all probabilities are the same, the second and third models are the same. In this case, $\gamma = -(C/2)\ln(1 - 2/C)$. If $C \ge 2$, the bounds for the two coupon collector problems are essentially the same.

It is also of interest to determine the nature of the bounds on T_3 when the mass of the distribution is concentrated on just a couple of points. For example, when the mass is centered equally on two points, that is, $p_0 = p_1 = 1/4$, $\gamma = 2\ln 4/3$ and the bounds on T_3 continue to grow as $C \ln(C/P_{cyclic}(T_3, C))$.

This last observation demonstrates that it will be very difficult to collect all coupons when it is possible to accurately target balls in bins but still leave some small uncertainty regarding their actual locations, as shown for the first model.

6 Conclusions

We have examined three models for the placement of high-K dielectric regions in the mask-based decoder when w is the minimal width and separation of these regions measured in the pitch of NWs. The first model assumes that lithographic regions have a small amount of random variation in their relative locations on masks and large displacement of each mask. The second model has random placement of region boundaries. The third assigns a constant number of boundaries to the space between every two NWs but perturbs their nominal location to obtain their actual location.

The number of MWs, M, needed to control N NWs with probability $1 - \epsilon$ is smallest for the first model for which $M = 2\log_2 N - 2\log_2 w + (2w)\log_2(w/(1-\epsilon))$. We find it surprising that in this model M is minimized by choosing to place LRs on separate masks rather than aggregate them on one mask. When $w = 3$, between $2\log_2 N + 46$ and $2\log_2 N + 53$ are required.

As NW sizes continue to shrink, the third model better illustrates the number of MWs needed to control N undifferentiated NWs. In this case the number of MWs is at least $2\log_2 N + \frac{C}{\gamma}\ln\left(\frac{C}{2\epsilon(1+\epsilon)}\right)$ where $1 \leq \gamma \leq 1.25$.

As these numbers suggest, the mask-based decoder is impractical unless the NWs on which it is used are very long allowing for N, the number of NWs, to be very large. Each NW in an $N \times N$ crossbar must be long enough to accommodate N perpendicular NWs as well as M MWs. Fortunately, SNAP NWs can be millimeters in length, which would allow for large values of N.

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