Analysis of Mask-Based Nanowire Decoders

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This research was funded in part by NSF Grant CCF-0403674. JE Savage was also supported in part by École Polytechnique. An early version of this paper appeared in ISVLSI 2005.

Abstract

Stochastically assembled nanoscale architectures have the potential to achieve device densities 100 times greater than today's CMOS. A key challenge facing nanotechnologies is controlling parallel sets of nanowires, such as those in crossbars, using a moderate number of mesoscale wires. Three methods have been proposed to control NWs using a set of perpendicular mesoscale wires. The first is based on nanowire differentiation during manufacture, the second makes random connections between nanowires and mesoscale wires, and the third, a mask-based approach, interposes high-K dielectric regions between nanowires and mesoscale wires. All three addressing schemes involve a stochastic step in their implementation. In this paper we analyze the mask-based approach and show that a large number of mesoscale control wires is necessary for its realization.

Index Terms: Emerging technologies, memory structures, stochastic processes

I. Introduction

The crossbar, a simple but well-known connection network, consists of two orthogonal sets of parallel wires (see Figure 1). Switches are positioned at the crosspoints defined by the intersections of pairs of wires. Crossbars can be used as switching networks, memories, and programmed logic arrays.

Chemists have developed methods to assemble nanowires (NWs) into crossbars [1], [2], [3], [4]. They have realized switches by placing a thin layer of bistable molecules, such as rotaxanes or [2]-catenanes, between two orthogonal sets of NWs [5], [6], [7]. When a large positive or negative electric field is applied between two orthogonal NWs, the molecules at their crosspoint become either conducting or nonconducting. A smaller electric field can then measure the conductivity of the crosspoint without changing it.

A number of methods have been devised to produce NWs using vapor-liquid-solid (VLS) processes [8], [9], nanoimprinting [10], superlattice NW pattern transfer (SNAP) [2], and nano-lithography [11], [3]. NWs produced through VLS can be differentiated. They can be grown with different electrical or chemical properties before being stochastically assembled into crossbars. NWs produced by the other three methods are undifferentiated.

Both differentiated and undifferentiated NWs must interface with larger mesoscale technology. An important challenge is to control individual NWs with mesoscale wires (MWs) without losing the high crosspoint density NWs allow. This challenge can be met by a) positioning MWs at right

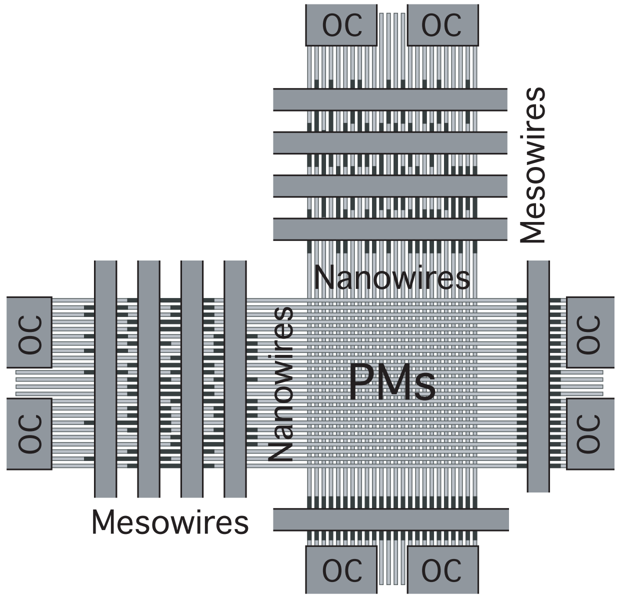

Fig. 1. A crossbar formed from two orthogonal sets of NWs with programmable molecules (PMs) at crosspoints defined by intersecting orthogonal NWs. NWs are segmented into contact groups connected to pairs of ohmic contacts (OCs). To activate a NW in one dimension, a contact group is activated and MWs are used to deactivate all but one NW in that group. Data is stored at a crosspoint by applying a large electric field across it. Data is sensed with a smaller field after disconnecting OCs using the two additional MWs.

angles to the NWs and b) using MWs to apply electric fields to lightly doped regions of NWs. The application of an electric field to an exposed lightly doped region drives the conductance of that NW low. (See Fig. 2.) In other words, NWs combined with MWs form field effect transistors (FETs). Turning on a subset of the MWs turns off some subset of the NWs.

A **decoder** is a circuit that addresses (leaves conducting) one NW (or a desired subset of NWs) by associating it with some subset of MWs. As explained in Section II, the following three methods have been proposed to control NWs with MWs: a) grow differentiated NWs containing lightly doped regions then place a random subset of the NWs on a chip using fluidic self-assembly [13], [14]; b) make random contacts between MWs and undifferentiated NWs [15], [16]; and c) randomly place lithographically defined high-K dielectric regions between MWs and undifferentiated lightly doped NWs [17].

The number of MWs needed to control N NWs with high probability has been determined analytically for the first [18], [19], [20] and second method [15], [20], [16]. Here we analyze the third method, which is implemented using a randomized mask-based decoder. We show that

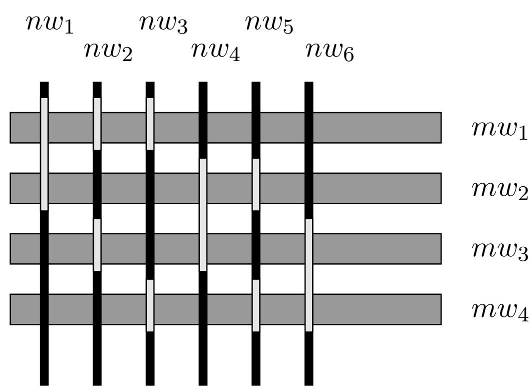

Fig. 2. A method for addressing six differentiated modulation-doped NWs $\{nw_1, \ldots, nw_6\}$ with four large mesoscale wires (MWs) $\{mw_1, \ldots, mw_4\}$. The lightly doped regions of each NW are highlighted. A NW is nonconducting if it has a doped region adjacent to a MW carrying a high electric field. If exactly two of the four MWs carry a high field, exactly one of the six NWs conducting. This idea is developed in [12].

for very reasonable assumptions it requires a large number of MWs.

Mask-based decoders (see Section II) are designed to work with undifferentiated NWs produced by nanoimprinting [10] or the SNAP process [2], [11]. These decoders use lithographically defined mesoscale rectangular regions of high-K dielectric (we call these **LR**s) to allow each MW to make some subsets of the NWs nonconducting (see Figure 3). Lithography, however, puts a lower limit on the size of such regions. The smallest regions must be randomly shifted to ensure all pairs of adjacent NWs are controlled by different subsets of MWs. This makes it possible to address individual NWs, but requires a large number of MWs, as we show.

A. Overview of the Paper

In Section II we describe three methods of addressing NWs with MWs: a) “encoded-NW decoders,” b) “randomized-contact decoders,” and c) “mask-based decoders.” In Section III we model the control MWs exert over NWs. We then focus specifically on masked-based decoders, giving a condition that LRs must satisfy in order for each NW to be addressed individually.

Methods for manufacturing undifferentiated NWs and mask-based decoders are described in Section IV. The limitations on photolithography that lead to uncertainties in the placement of LRs are examined and modeled probabilistically. This model is used in Section V to begin an analysis of the “randomized n-cycle mask-based decoder.” In this decoder, groups of adjacent NWs are connected to ohmic contacts. A mask-based decoder is then used to control individual

NWs within each group. Our goal is to determined how many MWs are required to individually address each NW.

In Section VI we present and analyze three models for the random placement of LRs that capture variation in LR placement as well as choices that a designer has in transferring LRs to a chip using one or several masks. These models are abstracted into three generalizations of the standard coupon collector problem: a) the "coupon collector problem with failures," b) the "targeted coupon collector problem," and c) the "multi-stage targeted coupon collector problem." In the standard coupon collector problem one of C coupons is selected with probability $1/C$ at each trial. The problem is to determine how many independent trials are required to collect all coupons with high probability. Here we consider more general probability distributions.

Drawing on the results in Section VI, the performance of the randomized n-cycle mask-based decoder is summarized in Section VII. Section VIII discusses several practical considerations for designing of mask-based decoders. Conclusions are drawn in Section IX.

II. Addressing NWs with MWs

This section describes three methods for addressing NWs with MWs. Each method assumes that NWs are divided into groups of approximately 10 adjacent NWs, called **contact groups**. Each contact group is connected to a separate ohmic contact. When a NW is addressed, a contact group is selected using standard CMOS circuitry, then all but one NW within the group is made nonconducting by turning on some subset of the MWs. All three methods of addressing NWs with MWs introduce uncertainty with regard to which MWs address which NWs. Thus, each method requires programmable circuitry to map external binary addresses to subsets of MWs.

A. *Differentiated NW Decoders*

Lieber *et al* have shown that differentiated NWs can be assembled into crossbars using fluidic methods [13], [14]. When NWs are manufactured using a VLS process, they can be grown with a pattern of lightly and heavily doped sections along their length [21], [22], [23], a process known as "modulation doping." Many copies of differently patterned NWs are collected in a large ensemble and deposited on a chip. As a result, the NWs in each contact group have doping patterns selected at random from the larger ensemble. NWs with distinct doping patterns can be individually addressed with MWs, as described in Fig. 2. This **encoded-NW decoder** is analyzed in [18], [24], [19].

Dehon *et al* show that all N NWs can be individually addressed more than 99% of the time using M MWs when $M \geq \lceil 2.2 \log_2 N \rceil + 11$ [18]. Three other addressing strategies are explored by Gojman *et al* [24], [19]. These include a) individually addressing half of the NWs in each contact group and b) addressing each NW doping pattern in at least p contact groups. Their analysis indicates that these two strategies require less area than the strategy that requires all NWs within each contact group be addressable [18]. A new technique for encoding of NWs through the use of shells of different types has also been proposed and shown to be competitive with modulation doping [25].

B. Randomized Contact NW Decoders

The **randomized-contact decoder** was proposed by Williams and Kuekes [15]. It controls N NWs with M MWs by making random contacts between them with probability of 1/2. They state that $M \geq 5 \log_2 N$ MWs suffice to provide unique addresses to all N NWs [15]. This method has been analyzed empirically and approximately by Hogg *et al* [16] who show that the probability that all NWs in a contact group are controllable rises rapidly to near 1 as M increases from slightly less than $4 \log_2 N$ to slightly more than $6 \log_2 N$. They also explore the number of MWs needed when contacts are imperfect. Rachlin and Savage have done a mathematical analysis of this model and derive bounds on the number of MWs needed to ensure that all NWs are controllable with probability at least $1 - \epsilon$ [20].

C. Mask-Based NW Decoders

The third decoder [26], [17], called a **mask-based decoder**, places lithographically-defined high-K dielectric regions in between MWs and lightly doped NWs. If an LR lies between a set of NWs and a MW, those NWs are made nonconducting by that MW when it carries a sufficiently strong electric field. Manufacturing constraints limit the precision with which LRs can be placed. These manufacturing constraints are described below.

An idealized **logarithmic mask-based decoder** is shown in Figure 3 (a) [27]. This decoder uses k pairs of MWs to control $N = 2^k$ NWs. For $1 \leq j \leq \log_2 N$ the two MWs in the j^{th} pair each lie over a row of 2^{j-1} evenly spaced LRs. These two rows of LRs cover complementary

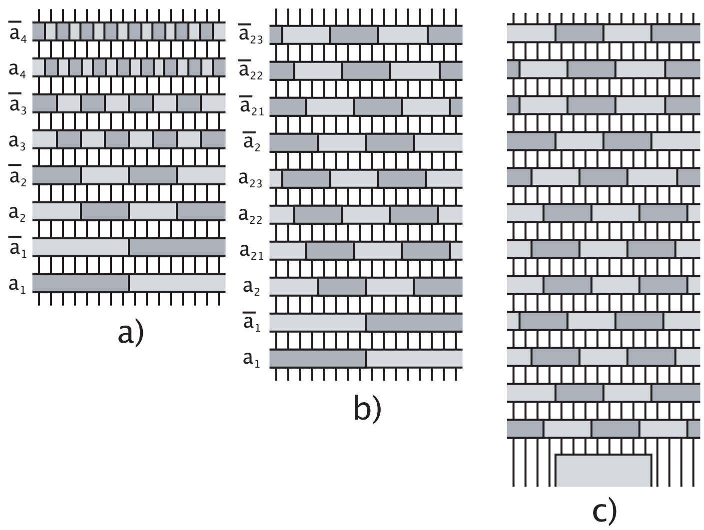

Fig. 3. Three mask-based decoders in which horizontal MWs lie across vertical NWs. The dark gray rectangles indicate the locations of high-K lithographically-defined dielectric regions (LRs) under the MWs. The regions under a MW determine which NWs become nonconducting when that MW carries an electric field. (a) A logarithmic mask-based decoder which uses $2 \log N$ MWs to select one of N NWs. (b) A hybrid mask-based decoder in which rows of the smallest manufacturable LRs are shifted and repeated to provide control over individual NWs. (c) A randomized mask-based decoder in which small groups of NWs are connected to an ohmic contact (bottom).

halves of the NWs (See Figure 3(a)). When a field is applied to one of the two MWs in a pair, exactly half of the NWs are turned off. Each pair of MWs turns off half of the NWs left on by the previous pair. This allows a logarithmic mask-based decoder to select exactly one NW to remain conducting when a field is applied to one MW in each of the k pairs of MWs. A logarithmic mask-based decoder thus assigns a unique address to each of the N undifferentiated NWs using $2 \log_2 N$ MWs.

Unfortunately, the logarithmic mask-based decoder isn't feasible. It requires that LRs have lengths that are equal to the pitch of NWs and that the position of their boundaries be tightly controlled, characteristics that cannot be met with lithography. As an alternative Beckman *et al* have proposed the **hybrid decoder** [17] to cope with this uncertainty. (See Figure 3 (b).) This decoder has linear and logarithmic portions.

The logarithmic portion is a logarithmic mask-based decoder that resolves the set of active NWs down to a small contact group of w NWs. In the linear portion of the hybrid decoder the goal is to have one LR left boundary and one LR right boundary fall in the space between each pair of NWs. If each LR has length exactly equal to w NW pitches, $2w$ rows of LRs would suffice to allow fields to be applied to MWs so that one NW in a set of w NWs is active and the rest inactive, a condition derived in Section III-A.

We refer to the $2w$ rows of evenly spaced LRs, where each row is offset by one NW pitch from the previous rows, as **one cycle**. Because this type of precision isn't possible at the nanometer scale, multiple cycles are needed to ensure that with high probability both left and right LR boundaries fall between pairs of NWs. (See Figure 3 (c).) We refer to these cycles as a **randomized n-cycle linear decoder**

Since the placement of LRs is difficult to control at the nanometer scale, it is more likely that the logarithmic portion of the mask-based decoder would be replaced by conventional lithographic-scale decoder in which contact groups of w NWs are connected to pairs of ohmic contacts and one contact group activated at a time by this decoder. A randomized n-cycle linear decoder is then created for each contact group, all of which share the same set of MWs. We call this a **randomized n-cycle mask-based decoder** and analyze its performance in Section VII.

III. Criteria for NW Addressability

In all of the decoders described in Section II, turning on a MW increases the resistance of some random subset of NWs. When multiple MWs are turned on, the increases in resistance introduced by each MW add. When a NW is addressed, its resistance must be much less than the resistance of all $w-1$ other NWs in the same contact group when combined in parallel.

When all MWs are turned off, let r_{low} denote the maximum resistance of any one NW. A MW **controls** a section of a NW if, when turned on, it increases the NW's total resistance by an amount much larger than wr_{low}. This ensures that when a MW is turned on, the combined resistance of the NWs it controls is greater than the resistance of the NW being addressed. The section of the NW under that MW is said to be **controllable**. Conversely, the section of a NW under a MW is **noncontrollable** if the MW increases a NWs resistance by an amount much less than r_{low}. Finally the section is **ambiguous** if it is neither controllable nor noncontrollable.

A MW **controls**, **does not affect**, or **is ambiguous with respect to** a NW if the NW has a section that is controllable, noncontrollable or ambiguous underneath that MW. A NW, n_i, is **individually addressable** if there exists some subset of MWs, S, such that every MW in S does not affect n_i and every other NW in the same contact group as n_i is controlled by at least one NW in S.

A. Conditions for NW Control in the Linear Decoder

In a mask-based decoder, the locations of the LRs determine which MWs control which NWs. Consider adjacent NWs n_a and n_b where n_a is to the left of n_b. If a LR under a MW has a left boundary between n_a and n_b, the section of n_a under the MW is uncontrollable and the section of n_b under the MW is controllable. As the LR's left boundary moves rightward there is a point at which the section of n_b goes from being controllable to ambiguous. Similarly, as the boundary moves leftward there is a point at which the section of n_a goes from being noncontrollable to ambiguous. The region between these two limits is called the **interNW region**. The following condition ensures that all pairs of NWs in a group of consecutive NWs are individually addressable.

Lemma III.1 *Assume that the length of and separation between LRs both span at least w NWs. All NWs in a group of w consecutive NWs are addressable if and only if the left boundary and right boundary of two different LRs fall in the interNW region associated with each of the $w-1$ pairs of consecutive NWs.*

Proof: A NW n_i is individually addressable if and only if there exist a subset of MWs, denoted S_i, such that no MW in S_i affects n_i and all $w-1$ other NWs are controlled by at least one MW in S_i.

For the "if" case assume all consecutive pairs of NWs have left and right LR boundaries in the interNW regions between them and consider an arbitrary NW n_a. There exists a MW m_1 that lies on top of a LR whose left boundary is in the interNW region to the right of n_a. Since the LR must have a length spanning at least w NWs, MW m_1 controls all NWs in question to the right of n_a. Similarly, there exists a MW m_2 that lies on top of a LR whose right boundary is in the interNW region to the left of n_a. This MW controls all the NWs in question to the left of n_a. The set $S_a = \{m_1, m_2\}$ individually addresses n_a.

For the "only if" case, assume all NWs are independently addressable. Consider any two adjacent NWs, n_a and n_b, where n_a is to the left of n_b and I_{ab} is the interNW region between them. If n_a is individually addressable, there must be a MW in S_a that controls n_b but not n_a. This implies that the LR under this MW has its left boundary in I_{ab}. Similarly, since n_b is individually addressable, there exists a MW that controls n_a but not n_b, and thus some LR has its right boundary in I_{ab} as well. ■

This lemma proves that w consecutive NWs are controllable when right and left LR boundaries lie in each of $w-1$ interNW regions. As explained in Section IV, LR boundaries are placed stochastically. Consequently, many rows of LRs are necessary to ensure that these conditions hold with high probability.

This closely resembles the classic **coupon collector problem** in which a random "coupon" (here an interNW region) is collected at each of T trials (here LRs). One then asks how large T must be for each coupon to be collected with high probability. It is well-known that T must be proportional to $C \ln C$. In Section VI we introduce variants of the coupon collector problem that are relevant to the randomized mask-based decoder.

IV. STOCHASTIC ASSEMBLY OF MASK-BASED DECODERS

The randomized mask-based decoder can be used to control any type of long, straight and uniformly-spaced, lightly doped semiconducting NWs. This decoder was first proposed for use with NWs produced by the superlattice nanowire pattern transfer method (SNAP) [2]. It can also be used with NWs grown by nanoimprinting [10], [7].

SNAP uses molecular beam epitaxy (MBE) to make a GaAs/AlGaAs superlattice from which the AlGaAs layer is etched back, creating a sawtoothed block face. Metal is deposited through evaporation on edges and pressed onto an adhesive layer on silicon. After the superlattice is removed, metallic NWs remain attached to the silicon. These metallic NWs are used as a nanometer-scale mask for a thin silicon layer residing on top of silicon oxide to produce silicon NWs [11]. SNAP has also been shown to produce very long (2-3*mm*), small (8-10*nm*), and largely defect-free NWs having a uniform pitch (16-60*nm*) that can be deposited on a chip with each application of SNAP

In more recent experiments [17] SNAP has been used to create an array of 150 silicon NWs with width 13*nm* and pitch 34*nm*s. To produce lightly doped NWs, the silicon is doped before metallic wire deposition and silicon NW etching. After exposing silicon NWs, a light etching is done to remove the top few nanometers so that the dopant concentration is reduced to a controllable level. Control over groups of consecutive of NWs was demonstrated using lithographically produced high-K dielectric regions.

A. LR Manufacture

To deposit LRs on a chip using lithography one or more masks are constructed containing multiple rectangular openings. When openings are first made in masks, a one-time process, the separation between the rectangles as well as their size can vary somewhat from their intended values. Additionally, when masks are used, it is difficult to control the precise alignment of the openings with the NWs. The offset of a mask from its intended location may be large.

After light is passed through the openings in a mask onto a photoresist, an etching process either removes the lithographically defined regions (positive photoresist) or their complement (negative photoresist). The duration of the etching process, which cannot be precisely controlled, causes variation in the length and width of the LRs.

Let $\boldsymbol{\rho}$ denote the pitch of the NWs. We refer to the intended location of a LR's right or left boundary, relative to the NWs, as its nominal location. Variation in mask manufacture, mask placement, and mask application, all cause a LR's endpoint to vary from its nominal location. In the absence of variation, $2w$ left and $2w$ right LR boundaries suffice to create a perfect 1-cycle linear decoder (see Lemma III.1). Variation, however, introduces the need for multiple cycles, which we assume are placed using one or more masks.

E-beam lithography is currently too expensive for mass production, but it sets a limit on the best possible conditions. Using it a) masks can be offset by 50 to 100*nm*s from their intended locations, b) the length and relative placement of rectangular mask openings can vary by 5 to 10 *nm*s from their intended locations on a mask, and c) etching of photoresist can increase the length of LRs by up to 5*nm*s on a chip [28]. If photolithography is used, the longer wavelength of the radiation results in larger variations in these parameters. Uncertainty in mask placement and variation in mask manufacture are independent of the type of lithography employed.

B. Modeling Variation in Mask Placement

Let $\boldsymbol{d_{off}}$ **be the offset of a mask from its ideal location**, which we assume places the nominal locations of LR boundaries at the midpoint between NWs. d_{off} is defined in terms of the location of a particular but arbitrary LR boundary that we call the **canonical LR boundary, $\mathbf{LR_0}$**. (See Figure 4.) If d_{off} can be large relative to a NW pitch ρ, as we assume is the case, then the assumed uniformity in the placement of NWs allows us to replace d_{off} by the **phase difference $\boldsymbol{\theta}$** which is restricted to the interval $-\rho/2 \leq \theta \leq \rho/2$. Note that $\theta = 0$ corresponds

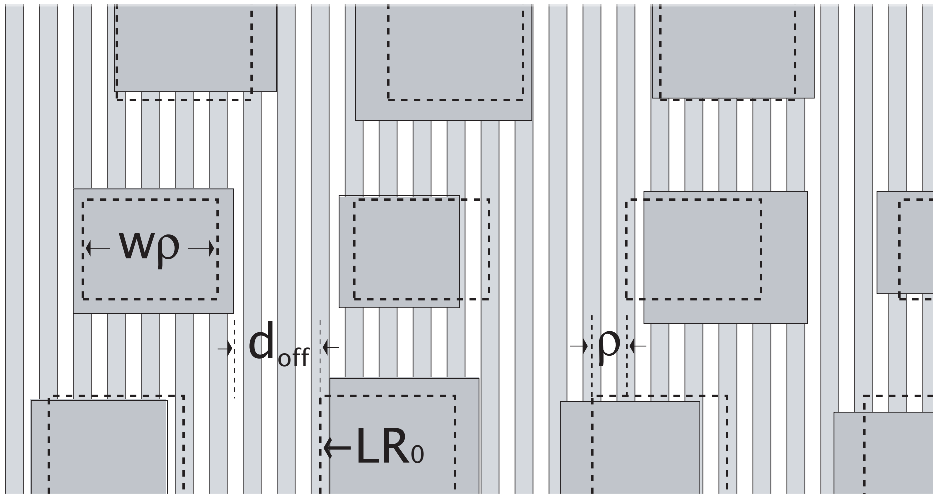

Fig. 4. Each LR has a nominal location on a mask indicated by dashed lines. Its actual location depends on random peturbances in endpoints denoted by random variables $\{d_k\}$. The location of a mask containing LRs is specified by relative offset d_{off} to a canonical LR denoted LR_0. Also note in this figure, $\theta = \rho/2$, $w = 4$, and full cycle of LRs would consist of $2w = 8$ rows. LEFT BOUNDARY OF LR_0 SHOULD BE IN THE SPACE BETWEEN NWs.

to the boundary LR_0 being at the middle of the space between two NWs. It is not important which two NWs it lies between.

Because we assume that the variation of d_{off} is large relative to ρ, **we model θ as a uniform random variable (r.v.) over the interval $[-\rho/2, \rho/2]$**. If the variation in d_{off} is small, as would be the case when the width and pitch of NWs is large, a non-uniform distribution in d_{off} would be appropriate, a case that we ignore.

C. Modeling Variation in LR Boundary Placement

When θ is fixed, uncertainties in LR boundary locations result from uncertainties in a) the inscribing of rectangles on masks, b) the exposure of photoresist by electromagnetic radiation through mask rectangles, and c) the photoresist etching time. We collect all these variations in a r.v., d, associated with each LR boundary. The actual location of a LR boundary is determined by θ, the offset of the nominal location of the boundary relative to the adjacent NWs, and d, the change in the position of the boundary relative to its nominal location.

We assume that d has a symmetric probability distribution $f(d)$ that decreases monotonically with d from $d = 0$. This reflects the fact that small variations in d are expected and variations are equally likely to be positive or negative. We also assume that the r.v.s $\{d_k\}$ associated with LR left and right boundaries are statistically independent and identically distributed.

Lemma III.1 states that for all NWs to be controllable, a LR right and left boundary must fall in each interNW region between each pair of consecutive NWs. If a LR boundary does not fall into an interNW region, the LR boundary is said to **fail**. If a boundary does not fail, it may fall in the interNW region closest to its nominal location, or some other interNW region. We refer to the interNW region closest to the nominal location as the **targeted interNW region**

For each LR boundary we let $p_i(\theta)$ be the **probability, given a mask phase difference of θ, that a LR boundary moves i regions to the right (left) from its targeted interNW region, when i is positive (negative).** Because the r.v.s $\{d_k\}$ are statistically independent when θ is fixed, the conditional joint probability that LR boundaries on a given mask fall into particular interNW regions is the product of the $p_i(\theta)$.

The facts cited in Section IV-A suggest that a LR boundary will vary by at most a few NW pitches when the mask offset d_{off} is fixed. That is, $q_i(\theta)$ will be non-zero only for small absolute values of i. We assume, $p_i(\theta) = 0$ for $i \geq w$. Since the right (and left) boundaries of LRs under the same MW are separated by $2w\rho$, only one such boundary has a nonzero probability of landing in any particular interNW region.

D. Additional Sources of LR Boundary Variation

LRs can also be placed using a stamping process [28]. The LRs in a stamp could then be inscribed using E-beam lithography and the stamp used multiple times. Two issues arise in the use of a stamp, a) uncertainties in the length and separation of LRs grow with the number of stampings and b) large uncertainties arise in the angular orientation of a stamp relative to NWs. It is estimated that the latter could be as large as 20 to 30 degrees. E-beam lithography may also introduce a small amount of angular uncertainty.

We do not explicitly model either the degradation of stamps nor the angular uncertainty introduced by both stamping and E-beam lithography in this paper. We believe, however, that these sources of variation can still be analyzed using our methods. Both have the effect of increasing the length of LR, and decreasing the amount of space between NWs. As a result, the width of an interNW region shrinks because sections of NWs that would otherwise be noncontrollable become ambiguous. This in turn reduces each $p_i(\theta)$.

V. Analyzing the Randomized Mask-Based Decoder

The randomized n-cycle and mask-based decoder uses a standard CMOS decoder to activate a contact group of w NWs. The high-K dielectric regions are then used to turn off all but one NW in a group. As described in Section II-C the regions are arranged in n cycles where a cycle requires $2w$ MWs. The randomized n-cycle decoder is designed to activate one of w NWs with high probability. As shown in Lemma III.1, this requires both a left and right LR boundary fall into each of the $w-1$ interNW regions.

During manufacture, the n cycles of the decoder are placed using some number of masks. Associated with each mask is a phase difference, θ. The θ's are uniformly distributed independent random variables. Given θ, we know the nominal positions of all LR boundaries produced by that mask. We assume that each LR boundary varies independently about its nominal position according to some unimodal symmetric distribution centered at 0.

We consider two models for assignment of cycles to masks. In the first, the **coarse-grained model**, we assume that the LRs under each MW are on separate masks. Thus, this model has $2nw$ different masks and one phase difference r.v. per mask, $\{\theta_t \mid 1 \leq t \leq 2nw\}$. In the second, the **fine-grained model**, we assume each mask places one or more cycles.

A randomized n-cycle mask-based decoder has N/w groups of w NWs. The decoder controls all N NWs if each NW in each set of w NWs is individually addressable. If there are m masks, let $\underline{\theta} = (\theta_1, \ldots, \theta_m)$ denote the set of m phase differences of these masks. For $1 \leq l \leq N/w$, let $F_l(\underline{\theta})$ denote the **failure to control all w NWs associated with the lth set of NWs given a value for $\underline{\theta}$**. Let $F(\underline{\theta})$ be the event that some NW in some set of N/w NWs is not controllable. It follows that $F(\underline{\theta})$ is the union of the events $F_l(\underline{\theta})$, $1 \leq l \leq N/w$. That is,

$$F(\underline{\theta}) = F_1(\underline{\theta}) \cup \cdots \cup F_{N/w}(\underline{\theta})$$

The unconditional probability of failure to control all N NWs, $P_r(F)$, is the average of $P_r(F(\underline{\theta}))$ over all the m values of the phase difference.

$$P_r(F) = \left(\frac{1}{\rho}\right)^m \int_{-\rho/2}^{\rho/2} \cdots \int_{-\rho/2}^{\rho/2} P_r(F(\underline{\theta}))\, d\theta_1 \cdots d\theta_m$$

Below we use the principle of inclusion and exclusion to bound $P_r(F)$.

Theorem V.1 *The probability $P_r(F)$ has the following bounds when $Q \leq 1/2$.*

$$Q(1 - Q/2) < P_r(F) \leq Q$$

where $Q = \rho^{-m}(N/w) \int_{-\rho/2}^{\rho/2} \cdots \int_{-\rho/2}^{\rho/2} P_r(F_1(\underline{\theta}))\, d\theta_1 \cdots d\theta_m$.

Proof: When the principle of inclusion and exclusion is used, the conditional probability $P_r(F(\underline{\theta}))$ has the following bounds.

$$Q(\underline{\theta}) - \sum_{l<m} P_r(F_l(\underline{\theta}) \cap F_m(\underline{\theta})) \leq P_r(F(\underline{\theta})) \leq Q(\underline{\theta})$$

Here $Q(\underline{\theta}) = \sum_{l=1}^{N/w} P_r(F_l(\underline{\theta}))$. Because the conditioned events $F_l(\underline{\theta})$ are assumed to be statistically independent, $P_r(F_l(\underline{\theta}) \cap F_m(\underline{\theta})) = P_r(F_l(\underline{\theta}))P_r(F_m(\underline{\theta}))$.

Let Q be the average of $Q(\underline{\theta})$, that is, $Q = \rho^{-m} \int_{-\rho/2}^{\rho/2} \cdots \int_{-\rho/2}^{\rho/2} Q(\underline{\theta})\, d\theta_1 \cdots d\theta_m$. Because the events $F_l(\underline{\theta})$ are identically distributed, $Q = (N/w)\overline{P_r(F_1(\underline{\theta}))}$ where $\overline{P_r(F_1(\underline{\theta}))}$ is defined below.

$$\overline{P_r(F_1(\underline{\theta}))} = \rho^{-m} \int_{-\rho/2}^{\rho/2} \cdots \int_{-\rho/2}^{\rho/2} P_r(F_1(\underline{\theta}))\, d\theta_1 \cdots d\theta_m$$

The sum in the above lower bound has $(N/w)(N/w-1)/2$ terms. Each term $P_r(F_l(\underline{\theta}))P_r(F_m(\underline{\theta}))$ is a product of statistically independent and identically distributed r.v.s. Thus, its average over $\underline{\theta}$ is $(N/w)(N/w-1)\left(\overline{P_r(F_1(\underline{\theta}))}\right)^2/2$. Because $Q = (N/w)\overline{P_r(F_1(\underline{\theta}))}$, this average becomes $((N/w-1)/(N/w))\, Q^2/2$ which is less than $Q^2/2$, giving the desired result. ■

Since the goal is to make Q very small, Q and $P_r(F)$ are very close. In the remainder of this paper we approximate the probability of failure to control all N NWs by Q.

Recall that $F_l(\underline{\theta})$ is the event that between every pair of w NWs we collect at least one left LR boundary and one right LR boundary given the phase differences $\underline{\theta}$. Let L (R) be the event that some left (right) LR boundary fails to be collected. Then $P_r(L \cup R)$ is the probability that one or the other type of boundary fails to be collected. It follows that

$$\max(P_r(L), P_r(R)) \leq P_r(L \cup R) \leq P_r(L) + P_r(R)$$

Lemma V.1 *The probability of a failure to collect both left LR and right LR boundaries between every pair of N NWs is within a factor of two of the probability of a failure to collect just left (or right) LR boundaries between every pair of N NWs.*

In light of the above fact, **we consider only the collection of left LR boundaries**.

In the next section we model the collection of LR left boundaries as variants of the coupon collector problem. When there is one mask for each LR under each MW, this problem is modeled by the coupon collector problem with failures (Section VI-A). When all LRs are produced by one mask, this is modeled by the targeted coupon collector problem (Section VI-B). In the final case when multiple cycles are produced by multiple masks, the problem is a multi-stage version of the latter problem (Section VI-C).

VI. Coupon Collection

In this section we analyze three increasingly general variants of the standard coupon collector problem: a) the coupon collector problem with failures, b) the targeted coupon collector problem, and c) the multi-stage targeted coupon collector problem. These generalizations are motivated by the cyclic placement of LRs in mask-based decoders. They are used in Section VII to analyze the randomized n-cycle mask-based decoder.

A. The Coupon Collector Problem with Failures

In the classic coupon collector problem, one of C coupons is randomly collected during each of T trials. Trials are independent and each coupon is selected with probability $1/C$. We introduce the **coupon collector problem with failure (CCF)** in which on each trial either a coupon fails to be collected with probability p_f (this models a LR boundary that falls outside of an interNW region) or a coupon is collected with probability $(1-p_f)/C$. T is chosen so that all coupons are collected with high probability.

Theorem VI.1 *Let Γ_{CCF} be the probability of failing to collect all C coupons in T trials when each trial has probability of failure $p_f = 1 - p_s$ and the probability of selecting the i^{th} coupon is $p_i = p_s/C$ for $1 \leq i \leq C$. Then, Γ_{CCF} and T satisfy the following bounds:*

$$z(1 - z/2) \leq \Gamma_{CCF} \leq z$$

where $z = C(1-p_s/C)^T$. Let $\phi_{CCF} = -C\ln(1-p_s/C)$. When z is small, minimizing z minimizes the bound on Γ_{CCC}. Then,

$$\frac{C}{\phi_{CCF}} \ln\left(\frac{C}{\Gamma_{CCF}(1+\Gamma_{CCF})}\right) \leq T \leq \frac{C}{\phi_{CCF}} \ln\left(\frac{C}{\Gamma_{CCF}}\right)$$

when $\Gamma_{CCF} \leq \sqrt{2}-1$. ϕ_{CCF} *satisfies* $p_s \leq \phi_{CCF} \leq p_s(1+p_s/C)$ *if* $C \geq 2$*. The bounds on* T *are minimized by maximizing* ϕ_{CCC}*.*

Proof: Theorem VI.1 is a special case of Theorem VI.2 below. When $p_r = p_s/C$ for all r, z and ϕ are the same as defined above. ■

B. The Targeted Coupon Collector Problem

We further generalize the coupon collector problem by allowing each trial to "target" a certain coupon. We call this the **targeted coupon collector problem**. As before, trials fail with probability p_f, but when a failure does not occur, each coupon is collected with a probability that is a function of the distance of the coupon from the targeted location. Let $p_0, p_1, ..., p_{C-1}$ be these probabilities. Clearly, $p_f + \sum_{r=0}^{C-1} p_r = 1$. The targeted coupon collector problem reduces to the coupon collector problem with failures when $p_r = p_s/C$ for all r.

Associated with each trial is a coupon t_j, $1 \leq j \leq T$, that is targeted. The probability that the j^{th} trial collects the i^{th} coupon is $p_{r(i,j)}$, where $r(i,j) = (i - t_j) \bmod C$. This has the effect of targeting the coupons in a cyclic fashion.

Consider C bins placed in a circle. At each of T trials, a ball is thrown from directly overhead. A trial collects the i^{th} coupon if it lands in the i^{th} bin. Each throw is aimed at a particular bin, t_j. The likelihood that a ball hits its target is always p_0. The probability that a ball deviates one bin to the right is p_1. The probability that a ball deviates one bin to the left is p_{C-1}. The probability that a ball fails to land in any bin at all is p_f. Clearly in this model, the probability that the ball hits a bin is independent of t_j.

As before, we wish to know how large T must be so that all coupons are collected with high probability. We are free to assign any value to each t_j, but we require these values to be chosen in advance. Each t_j cannot be based on the outcomes of previous trials. In our model we assume that each value of t_j is chosen an equal number of times and that T is a multiple of C. This is equivalent to cycling through all C coupons multiple times. Thus, we let $t_j = j \bmod C$ and call this the **cyclic** coupon collector problem (CCC).

Theorem VI.2 *Let* Γ_{CCC} *be the probability of failing to collect all* C *coupons in* T *trials,* T *a multiple of* C*, in the cyclic coupon collector problem when each trial has probability of failure* $p_f = 1 - p_s$ *and the probability of collecting the* i^{th} *coupon on the* j^{th} *trial is* $p_{r(i,j)}$*, where*

$r(i,j) = (i-j) \bmod C$. *Then,* Γ_{CCC} *and* T *satisfy the following bounds*

$$z(1-z/2) \leq \Gamma_{CCC} \leq z$$

where $z = C\prod_{r=0}^{C-1}(1-p_r)^{T/C} = Ce^{-\phi_{CCC}T/C}$ *and* $\phi_{CCC} = -\sum_{r=0}^{C-1}\ln(1-p_r)$. *When* z *is small, minimizing* z *minimizes the bound on* Γ_{CCC}. *Then,*

$$\frac{C}{\phi_{CCC}}\ln\left(\frac{C}{\Gamma_{CCC}(1+\Gamma_{CCC})}\right) \leq T \leq \frac{C}{\phi_{CCC}}\ln\left(\frac{C}{\Gamma_{CCC}}\right)$$

when $\Gamma_{CCC} \leq \sqrt{2}-1$. $p_s \leq \phi_{CCC} \leq p_s + \sum_{r=0}^{C-1} p_r^2$ *when* $p_r \leq .5$ *where* $p_s = \sum_{r=0}^{C-1} p_r$. *The bounds on* T *are minimized by maximizing* ϕ_{CCC}.

Proof: We use the principle of inclusion/exclusion. Let E_i be the event that i^{th} coupon is not collected after T trials and let $\Gamma_{CCC} = P(E_0 \cup \ldots \cup E_{C-1})$.

We assume that coupons are targeted in a cyclic fashion. Let E_i' be the event that the i^{th} coupon is not collected after C trials. The probability that the i^{th} coupon is not collected on the j^{th} trial is $(1-p_{r(i,j)})$, where $r(i,j) = (i-j) \bmod C$. In C consecutive trials, $r(i,j)$ will take on every value from 0 to $C-1$. Since trials are independent,

$$P(E_i') = \prod_{r=0}^{C-1}(1-p_r)$$

Now let E_i be the event that the i^{th} coupon is not collected in any of the T trials, T a multiple of C. Since $P(E_i) = P(E_i')^{T/C}$,

$$P(E_i) = \prod_{r=0}^{C-1}(1-p_r)^{T/C}$$

which is independent of i.

Now bound $P(E_h \cap E_i)$. Observe that the h^{th} and i^{th} coupons are not collected on the j^{th} trial with probability $(1-p_{r(h,j)}-p_{r(i,j)})$. Since $(1-a-b) \leq (1-a)(1-b)$, $(1-p_{r(h,j)}-p_{r(i,j)}) \leq (1-p_{r(h,j)})(1-p_{r(i,j)})$. As before, over C consecutive trials, $r(h,j)$ and $r(i,j)$ range over all values from 0 to $C-1$. Reordering terms allows us to write,

$$P(E_h \cap E_i) = P(E_h' \cap E_i')^{T/C} \leq \left[\prod_{r=0}^{C-1}(1-p_r)\prod_{r=0}^{C-1}(1-p_r)\right]^{T/C} = P(E_i)^2$$

Applying the principle of inclusion and exclusion we have that

$$\sum_{i=0}^{C-1} P(E_i) - \sum_{h<i} P(E_i)^2 \leq \Gamma_{CCC} \leq \sum_{i=0}^{C-1} P(E_i)$$

Since $\sum_{h<i} P(E_i)^2 \leq \left(\sum_{i=0}^{C-1} P(E_i)\right)^2 /2$, this yields the following bounds

$$z(1 - z/2) \leq \Gamma_{CCC} \leq z$$

where $z = \sum_{i=0}^{C-1} P(E_i)$. The inequality $z(1 - z/2) \leq \delta$ implies that $z \leq 1 - \sqrt{1 - 2\delta}$. In turn, this implies that $z \leq \delta(1 + \delta)$ when $\delta \leq \sqrt{2} - 1$. Thus, if $\Gamma_{CCF} \leq \sqrt{2} - 1$

$$\Gamma_{CCC} \leq z \leq \Gamma_{CCC}(1 + \Gamma_{CCC})$$

when $\Gamma_{CCC} \leq \sqrt{2} - 1$. Substituting in $z = C \prod_{r=0}^{C-1} (1 - p_r)^{T/C} = Ce^{-\phi_{CCC}T/C}$, where $\phi_{CCC} = -\sum_{r=0}^{C-1} \ln(1 - p_r)$, gives,

$$\frac{C}{\phi_{CCC}} \ln\left(\frac{C}{\Gamma_{CCC}(1 + \Gamma_{CCC})}\right) \leq T \leq \frac{C}{\phi_{CCC}} \ln\left(\frac{C}{\Gamma_{CCC}}\right).$$

Finally since $-x(1 + x) \leq \ln(1 - x) \leq -x$ when $x \leq .5$, $\sum_{r=0}^{C-1} p_r \leq \phi_{CCC} \leq \sum_{r=0}^{C-1}(p_r + p_r^2)$. Thus, $p_s \leq \phi_{CCC} \leq p_s + \sum_{r=0}^{C-1} p_r^2$ when $p_r \leq .5$ where $p_s = \sum_{r=0}^{C-1} p_r$. ■

It is of interest to know how sensitive the bounds on T are to the probability distribution $\{p_0, p_1, \ldots, p_{C-1}\}$. When all probabilities are the same, that is, $p_i = p_s/C$, the cyclic coupon collector problem is equivalent to the standard coupon collector problem with failure. In this case, $\phi_{CCC} = -C \ln(1 - p_s/C)$ and the bounds are the same.

Now consider a distribution that is far from uniform, one that is concentrated on just $C = 3$ points. If $p_0 = p_1 = p_2 = 1/4$ and $p_s = 3/4$, then $\phi_{CCC} = 3 \ln(4/3) = .86$. On the other hand, $\phi_{CCF} = -C \ln(1 - p_s/C) \approx p_s$ when $C \geq 10$ and $p_s \approx .5$. In this case, $\phi_{CCF} \approx \phi_{CCC}$ and the two bounds differ by a constant factor close to 1. Even if our ability to target specific coupons is good, the bounds on T continue to grow as $C \ln(C/\delta)$ where δ is the probability of failing to collect all coupons. Collecting all coupons remains difficult.

C. The Multi-Stage Targeted Coupon Collector Problem

The targeted coupon collector problem is now generalized to m "stages" where each stage captures the variation introduced by using a new mask. It is an extension of the cyclic coupon collection problem. In this problem, for some integer T_μ divisible by C, a **stage** is a set of T_μ trials where the j^{th} coupon, t_j, is targeted T_μ/C times. Associated with each stage is a uniformly distributed r.v. $\theta \in [-\rho/2, \rho/2]$ such that the probability of collecting a coupon targeted at a location i places away is $p_i(\theta)$, $0 \leq i \leq C-1$, a continuous function of θ. Also, $p_s(\theta) = 1 - p_f(\theta) = p_0(\theta) + \cdots + p_{C-1}(\theta)$ where $p_f(\theta)$ is the failure to collect any coupon on one trial. In addition, the stage r.v.s $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_m)$ are statistically independent. We call this the **multi-stage targeted coupon collector problem**.

Because this problem models an n-cycle randomized mask-based decoder, we are free to consider putting either one or multiple cycles on one stage. Thus, we would like to know how the failure probability $\Gamma_{MM} = P(E_0 \cup E_1 \cup \ldots \cup E_{C-1})$ depends on the number of cycles per stage. We show that it is smallest when each stage contains one cycle.

Theorem VI.3 *Let Γ_{MS} be the probability of failure to collect all coupons in the multi-stage targeted coupon collection problem with m stages in T trials when there are T_μ cycles in the μ^{th} stage, T_μ a multiple of C, $1 \leq \mu \leq m$, and $T = T_1 + \cdots T_m$ where the stage r.v.s $\underline{\theta}$ are statistically independent. Then, Γ_{MS} and T satisfy the following bounds.*

$$z(1 - z/2) \leq \Gamma_{MS} \leq z$$

where $z = Ce^{-\phi_{MS} T/C}$ and $\phi_{MS} = -\ln \prod_{\mu=1}^{m} \left(\frac{1}{\rho} \int_{-\rho/2}^{\rho/2} \left(\prod_{r=0}^{C-1} (1 - p_r(\theta_\mu)) \right)^{T_\mu/C} d\theta_\mu \right)$

$$\frac{C}{\phi_{MS}} \ln \left(\frac{C}{\Gamma_{MS}(1 + \Gamma_{MS})} \right) \leq T \leq \frac{C}{\phi_{MS}} \ln \left(\frac{C}{\Gamma_{MS}} \right)$$

when $\Gamma_{CCC} \leq \sqrt{2} - 1$.

When z is small, minimizing z (maximizing ϕ_{MS}) minimizes the bound on Γ_{MS}. The quantity z is minimized by placing each cycle in a separate stage in which case z satisfies the following bound.

$$z \geq C \left(\frac{1}{\rho} \int_{-\rho/2}^{\rho/2} \prod_{r=0}^{C-1} (1 - p_r(\theta_\mu))\, d\theta_\mu \right)^{T/C}$$

Proof: We use the principle of inclusion/exclusion in which E_i is the event that i^{th} coupon is not collected after T trials and we let $\Gamma_{MS} = P(E_0 \cup \ldots \cup E_{C-1})$.

We derive bounds on the failure event conditioned on the r.v.s $\underline{\theta}$, namely, $\Gamma_{MS}(\underline{\theta}) = P(E_0 \cup E_1 \cup \ldots \cup E_{C-1} \mid \underline{\theta})$ and then average the bounds over all values of $\underline{\theta}$.

Let E_i^μ be the event that the i^{th} coupon fails to be collected during T_μ trials in the μ^{th} stage. It follows that $E_i = E_i^1 \cap \cdots \cap E_i^m$ where $\{E_i^1, E_i^2, \ldots, E_i^m\}$ are statistically independent given the parameters $\underline{\theta}$. It follows that the conditional probabilities factor, as stated below.

$$P(E_i \mid \underline{\theta}) = P_r(E_i^1 \mid \theta_1) \cdots P(E_i^m \mid \theta_m)$$

To employ the principle of inclusion/exclusion we derive a bound on the conditional probability $P(E_h \cap E_i \mid \underline{\theta})$. Using the definition of these two events and the reasoning employed in the proof of Theorem VI.2 we have the following bound.

$$P(E_h \cap E_i \mid \theta_1, \theta_2, \ldots, \theta_m) \leq \prod_{\mu=1}^{m} P^2(E_i^\mu \mid \theta_\mu)$$

Here $P(E_i^\mu \mid \theta_\mu)$ is independent of i although it is dependent on θ_μ.

Averaging the bounds over $\underline{\theta}$ and applying the reasoning of the proof of Theorem VI.2, we have that $z(1 - z/2) \leq \Gamma_{MS} \leq z$ where

$$\begin{aligned} z = & \left(\frac{1}{\rho}\right)^m \int_{-\rho/2}^{\rho/2} \cdots \int_{-\rho/2}^{\rho/2} \sum_{i=1}^{C} P(E_i \mid \underline{\theta})\, d\underline{\theta} = \sum_{i=1}^{C} \prod_{\mu=1}^{m} \left(\frac{1}{\rho} \int_{-\rho/2}^{\rho/2} P(E_i^\mu \mid \theta_\mu)\, d\theta_\mu \right) \\ & = C \prod_{\mu=1}^{m} \left(\frac{1}{\rho} \int_{-\rho/2}^{\rho/2} \left(\prod_{r=0}^{C-1} (1 - p_r(\theta_\mu)) \right)^{T_\mu/C} d\theta_\mu \right) = C e^{-\phi_{MS} T/C} \end{aligned}$$

The latter result follows because $P(E_i^\mu \mid \theta_\mu)$ is independent of i.

A lower bound to z follows from a lower bound to $\frac{1}{\rho} \int_{-\rho/2}^{\rho/2} \left(\prod_{r=0}^{C-1} (1 - p_r(\theta_\mu)) \right)^{T_\mu/C} d\theta_\mu$. Holder's inequality is stated below where $1/p + 1/q = 1$ and $p, q \geq 1$.

$$\int_X |f(y)g(y)|\, dy \leq \left(\int_X |f(y)|^p\, dy \right)^{1/p} \left(\int_X |g(y)|^q\, dy \right)^{1/q}$$

Let $X = [-\rho/2, \rho/2]$, $f(y) = \left(\prod_{r=0}^{C-1} (1 - p_r(\theta_\mu)) \right)$ and $g(y) = 1/\rho$. Then, the inequality becomes the following.

$$\int_{-\rho/2}^{\rho/2} \frac{1}{\rho} f(y)\, dy \leq \left(\int_{-\rho/2}^{\rho/2} f(y)^p\, dy \right)^{1/p} \left(\int_{-\rho/2}^{\rho/2} \rho^{-q}\, dy \right)^{1/q} = \left(\int_{-\rho/2}^{\rho/2} \frac{1}{\rho} f(y)^p\, dy \right)^{1/p}$$

Here we have used the fact that $(1/q) - 1 = -1/p$. Consequently, when $p = T_\mu/C$

$$\frac{1}{\rho} \int_{-\rho/2}^{\rho/2} \left(\prod_{r=0}^{C-1} (1 - p_r(\theta_\mu)) \right)^{T_\mu/C} d\theta_\mu \;\geq \left(\frac{1}{\rho} \int_{-\rho/2}^{\rho/2} \prod_{r=0}^{C-1} (1 - p_r(\theta_\mu))\, d\theta_\mu \right)^{T_\mu/C}.$$

This implies the following lower bound to z.

$$z \geq C \prod_{\mu=1}^{m} \left(\frac{1}{\rho} \int_{-\rho/2}^{\rho/2} \prod_{r=0}^{C-1} \left(1 - p_r(\theta_\mu)\right) d\theta_\mu \right)^{T_\mu/C} = C \left(\frac{1}{\rho} \int_{-\rho/2}^{\rho/2} \prod_{r=0}^{C-1} \left(1 - p_r(\theta_\mu)\right) d\theta_\mu \right)^{T/C}$$

But this is the bound that applies when each cycle is placed on a separate stage. ■

VII. Performance of the Randomized n-Cycle Mask-Based Decoder

In this section we bound the number of MWs required to control all NWs in a randomized n-cycle contact group mask-based decoder. We consider two models for random placement of LRs a) the **course-grained model** in which each LR is placed independently using a separate mask, and a) **the fine-grained model** in which LRs are placed using masks that contain one or more cycle. The course grained-model provides a conservative upper bound on the number of MWs required to control all NWs with high probability. The fine-grained model provides an upper bound on the number of MWs required using more optimistic assumptions.

A. The Coarse-Grained Model

In the coarse-grained model each LR is placed on a separate mask. Since we assume that mask displacement can be at least 50-100*nm*, this is comparable to the number of NWs (which might be as small as ten but could be larger) that are expected to fall under the smallest LR. Thus, one can view the LR boundaries as equally likely to fall between any pair of w NWs. Only one of the two boundaries of a given LR falls within a set w NWs. Thus, we can treat each boundary displacement as a uniformly distributed r.v. because all of its variation is in the displacement of the mask.

The randomized n-cycle mask-based decoder activates one set of w NWs in a group and contains one linear decoder with n cycles for each group. Each linear decoder addresses one NW by deactivating all but one of these w NWs. Theorem V.1 provides tight bounds on the probability $P(F)$ that not all NWs can be addressed. This bound is the sum of the probabilities of a failure to have a NW be addressable in one or more of the N/w sets of w NWs. For each NW to be addressable a left and a right NW boundary must fall in the interNW region between every adjacent pair of NWs. We consider only the collection of left LR boundaries and incur a penalty of at most a factor of two, as explained at the end of Section V. The probability that there is a

LR left NW boundary between each pair of NWs is modeled by the coupon collector problem with failures. The probability of failure to collect all coupons is bounded in Theorem VI.1. We use the upper bound on T to obtain the following bound on the total probability of failure, $P(F_{cg})$ for the coarse-grained case.

Theorem VII.1 *The probability of a failure to address all NWs in the coarse-grained model, $P(F_{cg})$, satisfies $P(F_{cg}) \leq \epsilon$, when T, the number of MWs in the linear portion of the randomized mask-based decoder, is chosen as follows.*

$$T = \frac{w-1}{p_s} \ln \left[\left(\frac{2N}{\epsilon} \right) \left(\frac{w-1}{w} \right) \right]$$

The smallest value of T that satisfies $P(F_{cg}) \leq \epsilon$ is close this value when ϵ is small.

Proof: As shown in Section V, $P(F_{cg})$ is at most twice the sum of the probabilities of failing to collect all LR left boundaries in N/w sets of w NWs. That is, $P(F_{cg}) \leq 2(N/w)\Gamma_{CCF}$ where Γ_{CCF} is the probability of failure to collect $C = w - 1$ coupons when the i^{th} coupon is collected with probability $p_i = p_s/C$ and $p_s = 1 - p_f$ where p_s and p_f are the probabilities of success and failure in collecting coupons. If T is chosen so that $\Gamma_{CCF} = (\epsilon w)/(2N)$, then $P(F_{cg}) \leq \epsilon$. We use the bounds of Theorem VI.1 to bound T when $\Gamma_{CCF} = (\epsilon w)/(2N)$. In particular, if $T = \frac{C}{p_s} \ln \left(\frac{C}{\delta} \right)$, $P(F_{cg}) \leq \epsilon$. By examining the steps in the approximations, it is clear that this bound is tight when ϵ is small. ■

Performance of the Model: The number T of MWs in the linear portion of the decoder to ensure that the probability of failing to address all N NWs in the coarse-grained model is very close to $((w-1)/p_s)\ln(2N/\epsilon)$ when $w \geq 10$, which is logarithmic in $2N$ with an additive term proportional to $-\ln \epsilon$. The denominator p_s is the probability that a LR boundary succeeds in falling into an interNW region. Because an interNW region is slightly more than the space between two NWs, $p_s \geq .5$. Hence $T \geq 2(w-1)\ln(2N/\epsilon)$.

Consider a concrete example in which there are $w = 10$ NWs per group, $N = 1,000$, and $\epsilon = .01$, that is, success is achieved in controlling all N NWs with probability .99 or higher. In this case, $T \geq 220$. This is a very large number of MWs.

This value for T should be compared to $T_{all.diff}$, the number of trials for the "all different" encoded-NW decoder described in [18] where it is shown that $T_{all.diff} \geq \lceil 2.2 \log_2 N \rceil + 11$ suffices to control N NWs with failure probability $\epsilon = .01$. When $w = 10$, $N = 1,000$,

$T_{all.diff} = 33$ MWs can control 1,000 NWs with probability .99. The method of [18] requires a very large number, C, of differently encoded NW types. In particular, C may be more than 10,000. This number can be greatly reduced with a small effect on T using decoding strategies analyzed in [19].

As these calculations illustrate, the randomized mask-based decoder for the coarse-grained model requires many more MWs to decode N NWs than other decoders when N is 1,000 or more. We now explore the case when multiple cycles are placed on one mask.

B. The Fine-Grained Model

In the fine-grained model several masks may be used. The mask phase differences are independent and uniformly distributed r.v.s. The displacement of LR boundaries are small and modeled by the multi-stage targeted coupon collection problem. As with the coarse-grained model, the problem of failure to address all N NWs, $P(F_{fg})$, is closely approximated by $2(N/w)\Gamma_{MS}$ where Γ_{MS} is the probability of failure to collect $C = w - 1$ coupons when the i^{th} coupon is collected with probability $p_i(\theta)$ on a mask with phase difference θ.

As shown in Theorem VI.3, the probability of failure to collect all coupons in the multi-state coupon collection problem is smallest when each cycle of MWs occurs in a different stage. We summarize the result below.

Theorem VII.2 *The probability of a failure to address all NWs in the fine-grained model satisfies, $P(F_{fg}) \leq \epsilon$, when T, the number of MWs in the linear portion of the randomized mask-based decoder, is chosen as follows.*

$$T = \frac{w-1}{\phi_{MS}} \ln \left[\left(\frac{2N}{\epsilon} \right) \left(\frac{w-1}{w} \right) \right]$$

where ϕ_{MS} is defined below.

$$\phi_{MS} = -\ln \left(\frac{1}{\rho} \int_{-\rho/2}^{\rho/2} \prod_{r=0}^{w-2} (1 - p_r(\theta_\mu))\, d\theta_\mu \right)$$

The smallest value of T that satisfies $P(F_{cg}) \leq \epsilon$ is close this value when ϵ is small.

Proof: As with the previous proof, we observe that $P(F_{cg})$ at most twice the sum of the probabilities of failing to collect all LR left boundaries in N/w sets of w NWs. That is, $P(F_{cg}) \leq 2(N/w)\Gamma_{MS}$ where Γ_{MS} is the probability of failure to collect $C = w-1$ coupons in the multi-stage coupon collection problem when the i^{th} coupon on a mask with phase difference θ is collected with probability $p_i(\theta)$. The bounds of Theorem VI.3 on Γ_{MS} are $z(1 - z/2) \leq \Gamma_{MS} \leq z$ where $z = (w-1)e^{-\phi_{MS}T/(w-1)}$. When z is small, ϕ_{MS} is approximated by z, which provides the desired result. ■

The bound on T for the fine-grained case is identical to that given for the coarse-grained model except that the denominator term p_s is replaced by ϕ_{MS}. Observe that ϕ_{MS} is increased and T decreased if the product term in the definition of ϕ_{MS} is reduced.

Lemma VII.1 *The factor ϕ_{MS} satisfies the following bound where $p_s(\theta)$ is the probability that a LR left boundary falls into an interNW region.*

$$\phi_{MS} \leq -ln \left(1 - \frac{1}{\rho} \int_{-\rho/2}^{\rho/2} p_s(\theta)\, d\theta \right)$$

Proof: The proof follows from the fact that $(1-a)(1-b) \geq (1-a-b)$. ■

Performance of the Model: Given that a mask is uniformly distributed, $\frac{1}{\rho}\int_{-\rho/2}^{\rho/2} p_s(\theta)\, d\theta$ is close to 1/2 if the width and space of NWs are equal to one half the NW pitch. Thus, ϕ_{MS} is close to $\ln 2 = .7$. Since p_s from the bound for the coarse-grained case is about .5, we conclude that the number of MWs required is approximately $(.5/.7) * 220 \approx 157$. The randomized mask-based decoder is inefficient even when the location of LR boundaries can be tightly controlled.

VIII. Practical Considerations

The mask-based decoder requires a large number of MWs to control all NWs with high probability. In this section we describe several practical considerations that may make mask-based decoding more attractive.

A. Address Translation Circuitry

To use a NW crossbar as a memory, each external binary address must be mapped to a different pair of orthogonal NWs. All three types of decoders described in Section II introduce uncertainty with regard which MWs address which NWs. As a result, programmable address translation circuitry (ATC) is required to map binary addresses to subsets of MWs.

When performing this mapping, we assume that each external binary address is divided into high and low order bits. Each of these binary sequences is used to separately address a NW along each dimension of the crossbar. ATC for each dimension maps the supplied binary sequence, B, to a contact group σ and a subset of MWs, $\mathcal{M}$. When the MWs in $\mathcal{M}$ are turned on, a NW in σ is addressed. The NW addressed by each B must be unique.

If each contact group has exactly w addressable NWs, the mapping from B to σ is fixed, it does not vary from decoder for decoder. Furthermore, if w is a power of 2, we can simply take σ to be the high order bits of B. For $\mathcal{M}$, however, we cannot use the low order bits of B. The subsets of MWs used to address individual NWs varies from contact group to contact group, and from decoder to decoder.

For each B, the ATC must store a value for $\mathcal{M}$. The number of bits required for each $\mathcal{M}$ is at most M, since any subset of M MWs can be specified using M bits. M bits are necessary if most of the 2^M subsets appear with approximately equal frequency. This holds for both differentiated NW decoders and randomized contact decoders, which use $\Omega(M) = \Omega(\log N)$ bits per address.

In mask-based decoders, however, each NW can be addressed using just two MWs, one MW to turn off all NWs to its left, the other to turn off all NWs to its right. Since each $\mathcal{M}$ is a subset of two MWs, it can be stored using $2\log(M) = \Omega(\log N)$ bits. Even though mask-based decoders require a large number of MWs, they do not require significantly larger ATC than other decoders.

B. Alternative Addressing Strategies

We have computed bounds on the number of MWs required so that every NW in every contact group is addressable with high probability $1 - \epsilon$. As explained in Section VII, this is equivalent to requiring any given contact group have all NWs addressable with probability approximately $\epsilon/(N/w)$. Here N/w, the number of contact groups, is on the order of 100.

As explained in [19] and [20], the number of MWs can be reduced if we relax the requirement that all NWs in all contact groups be addressable and modify the ATC accordingly. One approach is to require only most contact groups to have every NW addressable. If only a small number of contact groups fail to have every NW addressable, we can store each group that has failed in the ATC, and have it skip these groups when mapping binary addresses to contact groups.

This alternative addressing strategy is illustrated in the following example. Suppose every contact group has every NW addressable with probability $.955$. By computing the tail of a binomial distribution with $p = .955$ and $N = 100$, one can show that, with probability .99, no more than 10 of 100 contact groups fail. This only decreases the number of addressable NWs by a factor of 10 (from Nw to $0.9Nw$), but since each contact group need only have every NW addressable with probability $.045$, that is, $\epsilon = .045$, from the theorems in Section VII, the number of MWs is reduced by a factor of 2 (compare $\ln(2*9*100/.01)$ to $\ln(2*9/.045)$). This still implies that more than 70 MWs are required, which is significantly more than the number required by other decoding technologies when using the same addressing strategy. Under the same conditions less than 30 MWs are necessary when either an encoded-NW or randomized-contact decoder is used [19], [29].

IX. Conclusion

We have analyzed the randomized n-cycle mask-based decoder, a new method for addressing NWs by interposing lithographically defined high-K dielectric regions between NWs and MWs [17]. The process of placing LRs is stochastic due to two factors: a) the absolute location of masks relative to NWs is difficult to control and b) small random variations will occur in the relative placement of LRs relative to one another. We have created models for the stochastic assembly of this decoder to account for these variations.

We have established conditions that LR boundaries must satisfy to ensure that all NWs in a set of w NWs can be individually addressed, namely, both a LR right and left boundary must fall between each pair of NWs.

We have modeled the satisfaction of this condition as the collection of coupons in variants of the classical coupon collector problem. We have introduced three models, the *coupon collector problem with failures*, the *multi-stage targeted coupon collector problem*, and the *multi-stage multi-stage targeted coupon collector problem*. The first problem is the classical problem except that coupons may fail to be collected. The second is like the first except that over a series of trials each coupon is targeted the same number of times although nearby coupons may be collected instead. The third is the same as the second except that the trials are grouped into a series of stages wherein the probabilities associated with collecting coupons in a stage are parametrized with a different random variable for each stage. The coupon collector problems that we present are of interest in their own right and may be useful in studying problems unrelated to mask-based decoding.

When our bounds are converted into numerical values representing typical cases, we find that the randomized mask-based decoder requires almost an order of magnitude more MWs to address all NWs than the encoded-NW decoder. In both [19] and [20] it was demonstrated that relaxing the requirement that all NWs in all contact groups be addressable results in a substantial reduction in the number of MWs. Although this is also true for mask-based decoders, they still require significantly more MWs than either an encoded-NW or randomized-contact decoder.

A key lesson to be learned from these results is that it is difficult to individually address NWs if it is very likely that two adjacent NWs either are both controlled or both not controlled by any given MW. A strong correlation of this kind is a key characteristic of the randomized mask-based decoder. When NWs are differentiated before their random selection for deposition on a chip, this correlation disappears. A strong lack of correlation is also exhibited by the randomized contact decoder [15], [16], which requires fewer NWs.

X. Acknowledgement

The authors are pleased to acknowledge fruitful conversations with Benjamin Gojman.

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