A prevalent theme in the economics and computation literature is to identify natural price-adjustment processes by which sellers and buyers in a market can discover equilibrium prices. An example of such a process is tâtonnement, an auction-like algorithm first proposed in 1874 by French economist Walras in which sellers adjust prices based on the Marshallian demands of buyers, i.e., budget-constrained utility-maximizing demands. An dual concept in consumer theory is a buyer’s Hicksian demand, i.e., consumptions that minimize expenditure while achieving a desired utility level. In this paper, we identify the maximum of the absolute value of the elasticity of the Hicksian demand, i.e., the maximum percentage change in the Hicksian demand of any good w.r.t. the change in the price of some other good, as an economic parameter sufficient to capture and explain a range of convergent and non-convergent tâtonnement behaviors in a broad class of markets. In particular, we prove the convergence of tâtonnement at a rate of $O((1+\varepsilon^2)/T)$, in homothetic Fisher markets with bounded price elasticity of Hicksian demand, i.e., Fisher markets in which consumers have preferences represented by homogeneous utility functions and the price elasticity of their Hicksian demand is bounded, where $\varepsilon$ is the maximum absolute value of the price elasticity of Hicksian demand across all buyers. Our result not only generalizes known convergence results for nested CES Fisher markets, but extends them to mixed(nested CES markets and Fisher markets with continuous, non-concave, homogeneous utility functions. Our convergence rate covers the full spectrum of nested CES utilities, including Leontief and linear utilities, unifying previously existing disparate convergence and non-convergence results. In particular, for $\varepsilon = 0$, i.e., Leontief markets, we recover the best-known convergence rate of $O(1/T)$ for Leontief markets, and as $\varepsilon \to \infty$, i.e., linear Fisher markets, we obtain non-convergent behavior, as expected.
INTRODUCTION

Competitive (or Walrasian or market) equilibrium [4, 72], first studied by French economist Léon Walras in 1874, is the steady state of an economy—any market system governed by supply and demand [72]. Walras assumed that each producer in an economy would act so as to maximize its profit, while consumers would make decisions that maximize their preferences over their available consumption choices; all this, while perfect competition prevails, meaning producers and consumers are unable to influence the prices that emerge. Under these assumptions, the demand and supply of each commodity is a function of prices, as they are a consequence of the decisions made by the producers and consumers, having observed the prevailing prices. A competitive equilibrium then corresponds to prices that solve the system of simultaneous equations with demand on one side and supply on the other, i.e., prices at which supply meets demand. Unfortunately, Walras did not provide conditions that guarantee the existence of such a solution, and the question of whether such prices exist remained open until Arrow and Debreu’s rigorous analysis of competitive equilibrium in their model of a competitive economy in the middle of last century [4].

The Arrow-Debreu model comprises a set of commodities; a set of firms, each deciding what quantity of each commodity to supply; and a set of consumers, each choosing a quantity of each commodity to demand in exchange for their endowment [4]. Arrow and Debreu define a competitive equilibrium as a collection of consumptions, one per consumer, a collection of productions, one per firm, and prices, one per commodity, such that fixing equilibrium prices: (1) no consumer can increase their utility by unilaterally deviating to an alternative affordable consumption, (2) firms cannot increase profit by deviating to another production in their production set, and (3) the aggregate demand for each commodity (i.e., the sum of the commodity’s consumptions across all consumers) does not exceed to its aggregate supply (i.e., the sum of the commodity’s productions and endowments across firms and consumers, respectively), while the total value of the aggregate demand is equal to the total value of the aggregate supply, i.e., Walras’ law holds.

Arrow and Debreu proceeded to show that their competitive economy could be seen as an abstract economy, which today is better known as a pseudo-game [4, 35]. A pseudo-game is a generalization of a game in which the actions taken by each player impact not only the other players’ payoffs, as in games, but also their set of permissible actions. Arrow and Debreu proposed generalized Nash equilibrium as the solution concept for this model, an action profile from which no player can improve their payoff by unilaterally deviating to another action in the space of permissible actions determined by the actions of other players. Arrow and Debreu further showed that any competitive economy could be represented as a pseudo-game inhabited by a fictional auctioneer, who sets prices so as to buy and resell commodities at a profit, as well as consumers and producers, who respectively, choose utility-maximizing consumptions of goods in the budget sets determined by the prices set by the auctioneer, and profit-maximizing productions at the prices set by the auctioneer. The elegance of the reduction from competitive economies to pseudo-games is rooted in a simple observation: the set of competitive equilibria of a competitive economy is equal to the set of generalized Nash equilibria of the associated pseudo-game, implying that existence of competitive equilibrium in competitive economies as a corollary of the existence of generalized Nash equilibria in pseudo-games, whose proof is a straightforward generalization of Nash’s proof of the existence of Nash equilibria [59].

With the question of existence out of the way, this line of work on competitive equilibrium, which today is known as general equilibrium theory [57], turned its attention to questions of (1) efficiency,
(under what assumptions are competitive equilibria Pareto-optimal?) (2) uniqueness (under what assumptions are competitive equilibria unique?), and (3) stability (under what conditions would a competitive economy settle into a competitive equilibrium?). The first two questions were answered between the 1950s and 1970s [5, 7, 9, 11, 30, 33, 43, 63], showing that (1) under suitable assumptions (e.g., see Arrow [3]) competitive equilibrium demands are Pareto-optimal, and (2) competitive equilibria are unique in markets with an excess demand function, (i.e., the difference between the aggregate demand and supply functions), which satisfies the weak gross substitutes (WGS) condition (i.e., the excess demand of any commodity weakly increases if the price of any other commodity weakly increases, fixing all other prices). In regards to the question of stability, most relevant work is concerned with the convergence properties of a natural auction-like price-adjustment process, known as tâtonnement, which mimics the behavior of the law of supply and demand, updating prices at a rate equal to the excess demand [6, 49]. Research on tâtonnement in the economics literature is motivated by the fact that it can be understood as a plausible explanation of how prices move in real-world markets. Hence, if one could prove convergence in all exchange economies, then perhaps it would be justifiable to claim real-world markets would also eventually settle at a competitive equilibrium.

Walras conjectured, albeit without conclusive evidence, that tâtonnement would converge to a competitive equilibrium. While a handful of results guarantee the convergence of tâtonnement under mathematical conditions without widely agreed-upon economic interpretations [60, 70], Arrow and Hurwicz [7, 8] were the first to formally establish the convergence of tâtonnement to unique competitive equilibrium prices in a class of economically well-motivated competitive economies, namely those that satisfy the WGS assumption. Following this promising result, Scarf [66] dashed all hope that tâtonnement would prove to be a universal price-adjustment process that converges in all economies, by showing that competitive equilibrium prices are unstable under tâtonnement dynamics in his eponymous competitive economy without firms, and with only three goods and three consumers with Leontief preferences, i.e., the Scarf exchange economy. Scarf’s negative result seems to have discouraged further research by economists on the stability of competitive equilibrium [36]. Despite research on this question coming to a near halt, one positive result was achieved, on the convergence of a non-tâtonnement update rule known as Smale’s process [44, 50, 68, 71], which updates prices at the rate of the product of the excess demand and its inverse, in most competitive economies, even beyond WGS, again suggesting the possibility that real-world economies could indeed settle at a competitive equilibrium.

Nearly half a century after these seminal analyses of competitive economies, research on the stability of competitive equilibrium is once again coming to the fore, this time in computer science, perhaps motivated by applications of algorithms such as tâtonnement to load balancing over networks [46], or to pricing of transactions on cryptocurrency blockchains [52, 54, 65]. A detailed inquiry into the computational properties of market equilibria was initiated by Devanur et al. [31, 32], who studied a special case of the Arrow-Debreu competitive economy known as the Fisher market [17]. This model, for which Irving Fisher computed equilibrium prices using a hydraulic machine in the 1890s, is essentially the Arrow-Debreu model of a competitive economy, but there are no firms, and buyers are endowed with only one type of commodity, an artificial currency [17, 61]. Devanur et al. [31] used the Eisenberg-Gale convex program to solve Fisher markets for buyers with linear utility functions, thereby providing a (centralized) polynomial-time algorithm for equilibrium computation in these markets [31, 32]. Their work was built upon by Jain et al. [45], who extended the Eisenberg-Gale program to all Fisher markets in which buyers have continuous, quasi-concave, and homogeneous utility functions, and proved that the equilibrium of Fisher markets with such buyers can be computed in polynomial time by interior point methods.
Concurrent with this line of work on computing competitive equilibrium using centralized methods, a line of work on devising and proving convergence guarantees for decentralized price-adjustment processes (i.e., iterative algorithms that update prices according to a predetermined update rule) developed. This literature has focused on devising natural price-adjustment processes, like \textit{tâtonnement}, which might explain or imitate the movement of prices in real-world markets. In addition to imitating the law of supply and demand, \textit{tâtonnement} has been observed to replicate the movement of prices in lab experiments, where participants are given endowments and asked to trade with one another [39]. Perhaps more importantly, the main premise of research on the stability of competitive equilibrium in computer science is that for competitive equilibrium to be justified, not only should it be backed by a natural price-adjustment process as economists have long argued, but it should also be computationally efficient [61].

The first result on this question is due to Codenotti et al. [26], who introduced a discrete-time version of \textit{tâtonnement}, and showed that in exchange economies that satisfy WGS, the \textit{tâtonnement} process converges to an approximate competitive equilibrium in a number of steps which is polynomial in the approximation factor and size of the problem. Unfortunately, soon after this positive result appeared, Papadimitriou and Yannakakis [62] showed that it is impossible for a price-adjustment process based on the excess demand function to converge in polynomial time to a competitive equilibrium in general, ruling out the possibility of Smale’s process (and many others) justifying the notion of competitive equilibrium in all competitive economies. Nevertheless, further study of the convergence of price-adjustment processes such as \textit{tâtonnement} under stronger assumptions, or in simpler models than full-blown Arrow-Debreu competitive economies, remains worthwhile, as these processes are being deployed in practice [46, 52, 54, 65].

Toward this end, in this paper we make strides towards analyzing the computational complexity of discrete-time \textit{tâtonnement} in homothetic Fisher markets, i.e., Fisher markets in which consumers have continuous and homothetic preferences. An important concept in consumer theory is a buyer’s Hicksian demand, i.e., consumptions that minimize expenditure while achieving a desired utility level. In this paper, we identify the maximum elasticity of the Hicksian demand, i.e., the maximum percentage change in the Hicksian of any good w.r.t. the change in the price of some other good, as an economic parameter sufficient to capture and explain a range of convergent and non-convergent \textit{tâtonnement} behaviors in a broad class of markets. In particular, we prove the convergence of \textit{tâtonnement} in homothetic Fisher markets with bounded elasticity of Hicksian demand, i.e., Fisher markets in which consumers have preferences represented by homogeneous utility functions for which the elasticity of their Hicksian demand is bounded.

\textit{Interpretation via Pseudo-Game Theory.} Recall that in the pseudo-game associated with a competitive economy [4], a (fictional) auctioneer sets prices for commodities, firms choose what quantity of each commodity to produce, and consumers choose what quantity of each commodity to consume in exchange for their endowment. Running \textit{tâtonnement} in this pseudo-game amounts to the auctioneer running a first-order method, namely a gradient ascent dynamic on their profit function, while the consumers and firms reply with their best response. Interestingly, if the competitive economy satisfies WGS, then the excess demand function is monotone over all \textit{tâtonnement} trajectories. This pseudo-game can then be understood as a monotone variational inequality [35], whose solutions are competitive equilibrium prices.

As \textit{tâtonnement} seeks to be an explanation of real-world market behavior, proofs of its convergence have to ideally rely on justifiable economic assumptions that can be measured and confirmed.
via experiments. Using the connection between pseudo-games and competitive economies, the task of proving any theorem on the convergence of \( \text{tâtonnement} \) can then be reduced to discovering justifiable economic assumptions to impose so that the ensuing pseudo-game or variational inequality satisfies suitable mathematical conditions for convergence.

Unfortunately, without imposing significant additional assumptions (such as the excess demand function of the competitive economy being Lipschitz-smooth; see, for example, Golowich et al. [41]) or relying on more complex update rules such as extragradient descent [51], first-order methods are not guaranteed to converge in last iterates\(^3\) in monotone pseudo-games or monotone variational inequalities. Undeterred, in addition to assuming WGS, Cole and Fleischer [27] impose economic assumptions on the Marshallian own-price elasticity of demand and Marsallian income elasticity of demand, which imply Lipschitz-smoothness of the excess demand function over \( \text{tâtonnement} \) trajectories, and obtain convergence of \( \text{tâtonnement} \) in last iterates in Fisher markets with WGS and bounded price/income elasticity of Marshallian demand.

Paralleling the duality between WGS competitive economies and monotone pseudo-games, any homothetic Fisher market is equivalent to a zero-sum game. Moreover, this zero-sum game can be further reduced to a convex potential, i.e., the Eisenberg-Gale program’s dual, whose solutions are competitive equilibrium prices [22, 31]. With this equivalence in hand, we seek to identify economically justifiable assumptions that translate into mathematical conditions on the excess demand function that are sufficient for convergence, because first-order methods run on zero-sum games and convex potentials are otherwise not guaranteed to converge to an optimal solution in last iterates. One obvious candidate condition is Lipschitz-smoothness of the excess demand function, but this property does not hold in Leontief Fisher markets, a flavor of homothetic markets in which \( \text{tâtonnement} \) converges! Our research has led to the discovery that assuming bounded elasticity of Hicksian demand yields an excess demand function that is Lipschitz-continuous as well as Bregman-smooth w.r.t. to the KL divergence,\(^4\) two properties which are sufficient to guarantee the convergence of \( \text{tâtonnement} \) in homothetic Fisher markets. Our contribution, then, is to identify the economic assumptions that imply the requisite mathematical properties that yield convergence of \( \text{tâtonnement} \) in last iterates.

### 1.1 Technical Contributions

Earlier work [20, 22] has established a convergence rate of \( (1 - \Theta(1))^T \) for CES Fisher markets excluding the linear and Leontief cases, and of \( O(1/T) \) for Leontief and nested\(^5\) CES Fisher markets, where \( T \in \mathbb{N}_+ \) is the number of iterations for which \( \text{tâtonnement} \) is run. In linear Fisher markets, however, \( \text{tâtonnement} \) does not converge. We generalize these results by proving a convergence rate of \( O((1+\epsilon^2)/T) \), where \( \epsilon \) is the maximum absolute value of the price elasticity of Hicksian demand across all buyers. Our convergence rate covers the full spectrum of homothetic Fisher markets,

<table>
<thead>
<tr>
<th>Economy type</th>
<th>Pseudo-game type</th>
<th>Mathematical object</th>
</tr>
</thead>
<tbody>
<tr>
<td>WGS Economy</td>
<td>Monotone pseudo-game</td>
<td>Monotone Variational Inequality</td>
</tr>
<tr>
<td>Homothetic Fisher market</td>
<td>Zero-sum game</td>
<td>Convex Potential</td>
</tr>
</tbody>
</table>

Table 1. Summary of the equivalences among economy types, pseudo-game types, and mathematical objects.

\(^3\)We note that the standard convergence metric in the \( \text{tâtonnement} \) literature is convergence in last iterates (see, for example, Cheung et al. [22]).

\(^4\)Bregman-smoothness is a generalization of Lipschitz-smoothness introduced by Cheung et al. [24] following work by Birnbaum et al. [13]; see Section 2 for the mathematical definition.

\(^5\)See Chapter 10 of Cheung [20].
The convergence rates of tâtonnement for different Fisher markets. We color previous contributions in blue, and our contribution in red, i.e., we study homothetic Fisher markets where $\varepsilon$ is the maximum absolute value of the price elasticity of Hicksian demand across all buyers. We note that the convergence rate for WGS markets does not apply to markets where the price elasticity of Marshallian demand is unbounded, e.g., linear Fisher markets; likewise, the convergence rate for nested CES Fisher markets does not apply to linear or Leontief Fisher markets.

(b) Cross-price elasticity taxonomy of well-known homogeneous utility functions. There are no previously studied utility functions in the space of utility functions with negative Hicksian cross-price elasticity. Future work could investigate this space and prove faster convergence rates than those provided in this paper. We note that our convergence result covers the entire spectrum of this taxonomy (excluding limits of the $y$-axis).

Fig. 1. A summary of known results in Fisher markets.

including mixed markets with linear, Leontief, and (nested) CES utilities, unifying previously existing disparate convergence and non-convergence results. In particular, for $\varepsilon = 0$, i.e., Leontief Fisher markets, we recover the best-known convergence rate of $O(1/\tau)$, and as $\varepsilon \to \infty$, i.e., linear Fisher markets, we obtain the non-convergent behaviour of tâtonnement [29]. We summarize known convergence results in light of our results in Figure 1a.

We observe that, in contrast to general competitive economies, in homothetic Fisher markets, concavity of the utility functions is not necessary for the existence of competitive equilibrium (Theorem 2.2). A computational analog of this result also holds, namely that tâtonnement converges in homothetic Fisher markets, even when buyers’ utility functions are non-concave. Our results parallel known results on the convergence of tâtonnement in WGS markets, where concavity of utility functions is again not necessary for convergence [26].

1.2 Related Work

Following Codenotti et al.’s [26] initial analysis of tâtonnement in competitive economies that satisfy WGS, Garg and Kapoor [38] introduced an auction algorithm that also converges in polynomial time for linear exchange economies. More recently, Bei et al. [12] established faster convergence bounds for tâtonnement in WGS exchange economies.

Another line of work considers price-adjustment processes in variants of Fisher markets. Cole and Fleischer [27] analyzed tâtonnement in a real-world-like model satisfying WGS called the ongoing market model. In this model, tâtonnement once-again converges in polynomial-time [27, 28], and it has the advantage that it can be seen as an abstraction for in-market processes. Cole and Fleischer’s results were later extended by Cheung et al. [23] to ongoing markets with weak gross complements, i.e., the excess demand of any commodity weakly increases if the price of any other commodity weakly decreases, fixing all other prices, and ongoing markets with a mix of WGC and WGS.
commodities. The ongoing market model these two papers study contains as a special case the Fisher market; however Cole and Fleischer assume bounded own-price elasticity of Marshallian demand, and bounded income elasticity of Marshallian demand, while Cheung et al. assume, in addition to Cole and Fleischer’s assumptions, bounded adversarial market elasticity, which can be seen as a variant of bounded cross-price elasticity of Marshallian demand, from below. With these assumptions, these results cover Fisher markets with a small range of the well-known CES utilities, including CES Fisher markets with $\rho \in [0,1)$ and WGC Fisher markets with $\rho \in (-1,0]$.\footnote{We refer the reader to Section 2 for a definition of CES utilities in terms of the substitution parameter $\rho$.}

Cheung et al. [22] built on this work by establishing the convergence of tâtonnement in polynomial time in nested CES Fisher markets, excluding the limiting cases of linear and Leontief markets, but nonetheless extending polynomial-time convergence guarantees for tâtonnement to Leontief Fisher markets as well. More recently, Cheung and Cole [21] showed that Cheung et al.’s result extends to an asynchronous version of tâtonnement, in which good prices are updated during different time periods. In a similar vein, Cheung et al. [25] analyzed tâtonnement in online Fisher markets, determining that tâtonnement tracks competitive equilibrium prices closely provided the market changes slowly.

Another price-adjustment process that has been shown to converge to market equilibria in Fisher markets is proportional response dynamics, first introduced by Wu and Zhang for linear utilities [73]; then expanded upon and shown to converge by Zhang for all CES utilities [74]; and very recently shown to converge in Arrow-Debreu exchange economies with linear and CES ($\rho \in (0,1)$) utilities by Brânzei et al. [18]. The study of the proportional response process was proven fundamental when Birnbaum et al. [13] noticed its relationship to gradient descent. This discovery opened up a new realm of possibilities in analyzing the convergence of market equilibrium processes. For example, it allowed Cheung et al. [24] to generalize the convergence results of proportional response dynamics to Fisher markets for buyers with mixed CES utilities. This same idea was applied by Cheung et al. [22] to prove the convergence of tâtonnement in Leontief Fisher markets, using the equivalence between mirror descent [16] on the dual of the Eisenberg-Gale program and tâtonnement, first observed by Devanur et al. [32]. More recently, Gao and Kroer [37] developed methods to solve the Eisenberg-Gale convex program in the case of linear, quasi-linear, and Leontief Fisher markets.

An alternative to the (global) competitive economy model, in which an agent’s trading partners are unconstrained, is the Kakade et al. model of a graphical economies [48]. This model features local markets, in which each agent can set its own prices for purchase only by neighboring agents, and likewise can purchase only from neighboring agents. Under the WGS assumption, auction-like price-adjustment processes have been shown to converge in variants of this model [2].

2 PRELIMINARIES

\textbf{Notation.} We use caligraphic uppercase letters to denote sets (e.g., $\mathcal{X}$); bold lowercase letters to denote vectors (e.g., $\mathbf{p}$, $\mathbf{\pi}$); bold uppercase letters to denote matrices (e.g., $\mathbf{X}$) and lowercase letters to denote scalar quantities (e.g., $x$, $\delta$). We denote the $i$th row vector of a matrix (e.g., $\mathbf{X}$) by the corresponding bold lowercase letter with subscript $i$ (e.g., $x_i$). Similarly, we denote the $j$th entry of a vector (e.g., $\mathbf{p}$ or $x_i$) by the corresponding Roman lowercase letter with subscript $j$ (e.g., $p_j$ or $x_{ij}$).

If a correspondence $\mathcal{A} : \mathcal{X} \Rightarrow \mathcal{Y}$ is singleton valued for some $x \in \mathcal{X}$, for notational convenience, we treat it as an element of $\mathcal{X}$, i.e., $\mathcal{A}(x) \in \mathcal{X}$, rather than a subset of it, i.e., $\mathcal{A}(x) \subset \mathcal{X}$. We denote the set of numbers $\{1, \ldots, n\}$ by $[n]$, the set of natural numbers by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$, the set of non-negative real numbers by $\mathbb{R}_+$ and the set of strictly positive real numbers by $\mathbb{R}_{++}$. We denote the natural logarithm by $\log$. We let $\Delta_\epsilon = \{x \in \mathbb{R}_+^n \mid \sum_{i=1}^n x_i = 1\}$. We denote the interior of any set $\mathcal{A}$ by $\interior(A)$. We define $\mathcal{B}_\epsilon(x) = \{z \in \mathbb{Z} \mid ||z-x|| \leq \epsilon\}$ to be the closed epsilon ball centered
at $x$; here $\mathcal{Z}$ and $\|\cdot\|$ will be clear from context. We denote the partial derivative of a function $f : X \to \mathbb{R}$ w.r.t. $x_j$ at a point $x = y$ by $\partial_{x_j} f(y)$. We define the gradient $\nabla_x : C^1(X) \to C^0(X)$ as the operator which takes as input a functional $f : X \to \mathbb{R}$, and outputs a vector-valued function consisting of the partial derivatives of $f$ w.r.t. $x$. Finally, by notational overload, we define the subdifferential $g$ of a function $f$ at a point $a \in U$ by $\partial_x f(a) = \{g \mid f(x) \geq f(a) + g^T(x-a)\}$.

2.1 Mathematical Preliminaries

A function $f : \mathbb{R}^m \to \mathbb{R}$ is said to be homogeneous of degree $k \in \mathbb{N}_+$ if $\forall x \in \mathbb{R}^m, \lambda > 0, f(\lambda x) = \lambda^k f(x)$. Unless otherwise indicated, without loss of generality, a homogeneous function is assumed to be homogeneous of degree 1. Fix any norm $\|\cdot\|$. Given $\mathcal{A} \subset \mathbb{R}^d$, the function $f : \mathcal{A} \to \mathbb{R}$ is said to be $\lambda_f$-Lipschitz-continuous iff $\forall x_1, x_2 \in \mathcal{X}, \|f(x_1) - f(x_2)\| \leq \lambda_f \|x_1 - x_2\|$. If the gradient of $f$ is $\lambda_f$-Lipschitz-continuous, we then refer to $f$ as $\lambda_f$-Lipschitz-smooth.

2.2 Mirror Descent

Consider the optimization problem $\min_{x \in V} f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable convex function and $V$ is the feasible set of solutions. A standard method for solving this problem is the mirror descent algorithm [16]:

$$x(t + 1) = \arg\min_{x \in V} \{t_f(x, x(t)) + y_t \delta_h(x, x(t))\} \quad \text{for } t = 0, 1, 2, \ldots \quad (1)$$

$$x(0) \in \mathbb{R}^n \quad (2)$$

Here, $y_t > 0$ is the step size at time $t$, $t_f(x, y)$ is the linear approximation of $f$ at $y$, that is $t_f(x, y) = f(y) + \nabla f(y)^T(x-y)$, and $\delta_h(x, x(t))$ is the Bregman divergence of a convex differentiable kernel function $h(x)$ defined as $\delta_h(x, y) = h(x) - \ell_h(x, y)$ [19]. In particular, when $h(x) = \frac{1}{2} \|x\|^2$, $\delta_h(x, y) = \frac{1}{2} \|x - y\|^2$. In this case, mirror descent reduces to projected gradient descent [16]. If instead the kernel is the weighted entropy $h(x) = \sum_{i \in [n]} (x_i \log(x_i) - x_i + y_i)$, then the Bregman divergence reduces to the generalized Kullback-Leibler (KL) divergence [47]:

$$\delta_{KL}(x, y) = \sum_{i \in [n]} x_i \log \left( \frac{x_i}{y_i} \right) - x_i + y_i \quad (3)$$

which, when $V = \Delta_n$, yields the following simplified entropic descent update rule:

$$\forall j \in [m], \text{ } x_j^{(t+1)} = x_j^{(t)} \exp \left\{ -\frac{\partial_{x_j} f(x^{(t)})}{y_t} \right\} \quad \text{for } t = 0, 1, 2, \ldots \quad (4)$$

$$x_j^{(0)} \in \text{int}(\Delta) \quad (5)$$

A function $f$ is said to be $\gamma$-Bregman-smooth [24] w.r.t. a Bregman divergence with kernel function $h$ if $f(x) \leq t_f(x, y) + \gamma \delta_h(x, y)$. Birnbaum et al. [13] showed that if the objective function $f(x)$ of a convex optimization problem is $\gamma$-Bregman w.r.t. to some Bregman divergence $\delta_h$, then mirror descent with Bregman divergence $\delta_h$ converges to an optimal solution $f(x^*)$ at a rate of $O(1/t)$. We require a slightly modified version of this theorem, introduced by Cheung et al. [22], where it suffices for the $\gamma$-Bregman-smoothness property to hold only for consecutive pairs of iterates.

**Theorem 2.1 (Birnbaum et al. [13], Cheung et al. [22]).** Let $\{x_t\}_t$ be the iterates generated by mirror descent with Bregman divergence $\delta_h$. Suppose $f$ and $h$ are convex, and for all $t \in \mathbb{N}$ and for some $\gamma > 0$, it holds that $f(x^{(t+1)}) \leq t_f(x^{(t+1)}, x^{(t)}) + \gamma \delta_h(x^{(t+1)}, x^{(t)})$. If $x^*$ is a minimizer of $f$, then the following holds for mirror descent with fixed step size $\gamma$: for all $t \in \mathbb{N}$, $f(x^{(t)}) - f(x^*) \leq t/t \delta_h(x^*, x^{(0)})$. 

Denizalp Goktas, Amy Greenwald, Sadie Zhao

7
2.3 Consumer Theory

Let $X = \mathbb{R}_+^m$ be a set of possible consumptions over $m$ goods s.t. for any $x \in \mathbb{R}_+^m$ and $j \in [m]$, $x_j \geq 0$ represents the amount of good $j \in [m]$ consumed by consumer (hereafter, buyer) $i$. The preferences of buyer $i$ over different consumptions of goods can be represented by a preference relation $\succeq_i$ over $X$ such that the buyer (resp. weakly) prefers a choice $x \in X$ to another choice $y \in X$ if $x \succ_i y$ (resp. $x \succeq_i y$). A preference relation is said to be complete iff for all $x, y \in X$, either $x \succeq y$ or $y \succeq x$, or both. A preference relation is said to be transitive if, for all $x, y, z \in X$, $x \succeq y \succeq z$ whenever $x \succeq z$ and $y \succeq z$. A preference relation is said to be continuous if for any sequence $\{x^{(n)}, y^{(n)}\}_{n \in \mathbb{N}} \subset X \times X \left(x^{(n)}, y^{(n)} \rightarrow (x, y) \text{ and } x^{(n)} \succeq_i y^{(n)}\}$ for all $n \in \mathbb{N}$, it also holds that $x \succeq_i y$. A preference relation $\succeq_i$ is said to be locally non-satiated iff for all $x \in X$ and $\epsilon > 0$, there exists $y \in B_{\epsilon}(x)$ such that $y \succ_i x$. A utility function $u_i : \mathbb{R}^m \rightarrow \mathbb{R}^+_+\text{ assigns a positive real value to elements of } \mathbb{R}^m$, i.e., to every possible allocation of goods. Every continuous utility function represents some complete, transitive, and continuous preference relation $\succeq_i$ over goods s.t. if $u_i(x) \geq u_i(y)$ for two bundles of goods $x, y \in \mathbb{R}^m$, then $x \succeq_i y$.[6]

In this paper, we consider the general class of homothetic preferences $\succeq_i$ s.t. for any consumption $x, y \in X$ and $\lambda \in \mathbb{R}_+$, $x \succeq_i y \Rightarrow \lambda x \succeq_i \lambda y$ and $\lambda y \succeq_i \lambda x$, respectively. A preference relation $\succeq_i$ is complete, transitive, continuous, and homothetic iff it can be represented via a continuous and homogeneous utility function $u_i$ of arbitrary degree [6]. We note that any homogeneous utility function $u_i$ represents locally non-satiated preferences, since for all $\epsilon > 0$ and $x \in X$, there exists an allocation $(1 + \epsilon/\|x\|)x$ s.t. $u_i((1 + \epsilon/\|x\|)x) = (1 + \epsilon/\|x\|)u_i(x) > u_i(x)$, and $x - (1 + \epsilon/\|x\|)x \in B_{\epsilon}(x)$. The class of homogeneous utility functions includes the well-known Constant Elasticity of Substitution (CES) utility function family, parameterized by a substitution parameter $\rho_i \in [0, 1]$, and given by $u_i(x_i) = \rho_i \sum_{j \in [m]} a_{ij} x_{ij}^{\rho_i}$ with each utility function parameterized by the vector of valuations $a_i \in \mathbb{R}_+^n$, where each $a_{ij}$ quantifies the value of good $j$ to buyer $i$. CES utilities are said to be gross substitutes (resp. gross complements) if $\rho_i > 0 \text{ (if } \rho_i < 0)$. Linear utility functions are obtained when $\rho$ is 1 (goods are perfect substitutes), while Cobb-Douglas and Leontief utility functions are obtained when $\rho \to 0 \text{ and } \rho \to -\infty$ (goods are perfect complements), respectively:

| Linear: $u_i(x_i) = \sum_{j \in [m]} a_{ij} x_{ij}$ | Cobb-Douglas: $u_i(x_i) = \prod_{j \in [m]} x_{ij}^{a_{ij}}$ | Leontief: $u_i(x_i) = \min_{j: a_{ij} \neq 0} \frac{x_{ij}}{a_{ij}}$ |

Next, we define the consumer functions. The indirect utility function $v_i : \mathbb{R}_+^m \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ takes as input prices $p$ and a budget $b_i$ and outputs the maximum utility the buyer can achieve at that prices within that budget, i.e., $v_i(p, b_i) = \max_{x \in \mathbb{R}_+^m : p \cdot x \leq b_i} u_i(x)$.

The Marshallian demand is a correspondence $d_i : \mathbb{R}_+^m \times \mathbb{R}_+ \Rightarrow \mathbb{R}_+^m$ that takes as input prices $p$ and a budget $b_i$ and outputs the utility-maximizing allocations of goods at that budget, i.e., $d_i(p, b_i) = \arg \max_{x \in \mathbb{R}_+^m : p \cdot x \leq b_i} u_i(x)$.

The expenditure function $e_i : \mathbb{R}_+^m \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ takes as input prices $p$ and a utility level $v_i$ and outputs the minimum amount the buyer must spend to achieve that utility level at those prices, i.e., $e_i(p, v_i) = \min_{x \in \mathbb{R}_+^m : u_i(x) \geq v_i} p \cdot x$. If the utility function $u_i$ is continuous, then the expenditure function is continuous and homogenous of degree 1 in $p$ and $v_i$ jointly, non-decreasing in $p$, strictly increasing in $v_i$, and concave in $p$.

[6] Throughout this work, without loss of generality, we assume that complete, transitive, continuous, and homothetic preference relations are represented via a homogeneous utility function of degree 1, since any homogeneous utility function of degree $k$ can be made homogeneous of degree 1 without affecting the underlying preference relation by passing the utility function through the monotonic transformation $x \mapsto \frac{x}{k}x$. Denizalp Goktas, Amy Greenwald, Sadie Zhao
The **Hicksian demand** is a correspondence \( h_i : \mathbb{R}^m \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) that takes as input prices \( p \) and a utility level \( v_i \) and outputs the cost-minimizing allocations of goods at those prices and utility level, i.e., \( h_i(p, v_i) = \arg \min_{x \in \mathbb{R}^m} d_i(x) \geq v_i p \cdot x \). The Hicksian demand is convex-valued if the utility function is continuous and concave, and unique if the utility function is continuous and strictly concave [53, 55].

Fixing a buyer \( i \in [n] \), a good \( j \in [m] \) is said to be a **gross substitute** (resp. **gross complement**) for a good \( k \in [m] \setminus \{j\} \) if the Marshallian demand \( d_{ij}(p, b_i) \) is increasing (resp. decreasing) in \( p_k \). If \( d_{ij}(p, b_i) \) of buyer \( i \) for good \( j \) is instead weakly increasing (resp. decreasing), good \( j \) is said to be a **weak gross substitute** (WGS) (resp. **weak gross complement** (WGC)) for good \( k \).

When one considers the Hicksian demand instead, a good \( j \in [m] \) is said to be a **substitute** (resp. **complement**) for a good \( k \in [m] \setminus \{j\} \) if the Hicksian demand \( h_{ij}(p, b_i) \) is increasing (resp. decreasing) in \( p_k \). If \( h_{ij}(p, b_i) \) for good \( j \) is weakly increasing (resp. decreasing), good \( j \) is said to be a **weak substitute** (resp. **weak complement**) for good \( k \).

Finally, for any vector-valued function \( f : \mathbb{R}^m \rightarrow \mathbb{R}^m \), the **elasticity** \( \varepsilon_{f,i,x_j} : \mathbb{R}^m \rightarrow \mathbb{R} \) of output \( f_i(x) \) w.r.t. the \( j \)th input \( x_j \) evaluated at \( x = y \) is defined as \( \varepsilon_{f,i,x_j}(y) = \frac{\partial x_j f_i(y)}{f_i(y)} \). The **cross-price elasticity of Hicksian** (resp. **Marshallian**) **demand** for good \( j \) w.r.t. the price of good \( k \neq j \) at price \( p \) and budget \( b_i \) (resp. utility level \( v_i \)) is given as \( \varepsilon_{h,k,p}(p, v_i) (\varepsilon_{d,k,p}(p, b_i)) \). If \( k = j \), then we instead have the **Hicksian** (resp. **Marshallian** own-price elasticity of demand). We note that if the elasticity of Hicksian (resp. Marshallian) demand for good \( j \) is bounded w.r.t. the price of any good \( k \in [m] \), then it must be unique since demands must be differentiable\(^8\).

### 2.4 Fisher Markets

A **Fisher market** consists of \( n \) buyers and \( m \) divisible goods [17]. Each buyer \( i \in [n] \) has a budget \( b_i \in \mathbb{R}_+ \) and a utility function \( u_i : \mathbb{R}_+^m \rightarrow \mathbb{R} \). As is standard in the literature, we assume there is one unit of each good, and one unit of currency available in the market, i.e. \( \sum_{i \in [n]} b_i = 1 \) [61]. An instance of a Fisher market is given by a tuple \( (n, m, u, b) \), where \( u = (u_1, \ldots, u_n) \), and \( b \in \mathbb{R}_+^m \) is the vector of buyer budgets. We abbreviate as \((u, b)\), when \( n \) and \( m \) are clear from context.

An **allocation** \( X \) is a map from goods to buyers, represented as a matrix s.t. \( x_{ij} \geq 0 \) denotes the amount of good \( j \in [m] \) allocated to buyer \( i \in [n] \). Goods are assigned **prices** \( p \in \mathbb{R}_+^m \). A tuple \((X^*, p^*)\) is said to be a **competitive** (or **Walrasian**) **equilibrium** of a Fisher market \((u, b)\) if 1. buyers are utility maximizing constrained by their budget, i.e., \( \forall i \in [n], x_{ij}^* \in d_i(p^*, b_i) \); and 2. the market clears, i.e., \( \forall j \in [m], p_{j}^* > 0 \Rightarrow \sum_{i \in [n]} x_{ij}^* = 1 \) and \( p_{j}^* = 0 \Rightarrow \sum_{i \in [n]} x_{ij}^* \leq 1 \).

When the buyers’ utility functions in a Fisher market are all of the same type, we qualify the market by the name of the utility function, e.g., a linear Fisher market. A **mixed CES Fisher market** is a Fisher market which comprises CES buyers with possibly different substitution parameters. Considering properties of goods, rather than buyers, a (Fisher) market satisfies **gross substitutes** (resp. **gross complements**) if all pairs of goods in the market are gross substitutes (resp. gross complements). We define substitute Fisher markets and complements Fisher markets similarly. A Fisher market is **mixed** (gross) if all pairs of goods are either (gross) complements or (gross) substitutes. We refer the reader to Figure 1a for a summary of the relationships among various Fisher markets. A Fisher market is called **homothetic** if the buyers’ utility functions are continuous and homogeneous.

Given a Fisher market \((u, b)\), we define the **aggregate demand** correspondence \( q : \mathbb{R}_+^m \rightarrow \mathbb{R}^m \) at prices \( p \) as the sum of the Marshallian demand at \( p \), given budgets \( b \), i.e., \( q(p) = \sum_{i \in [n]} d_i(p, b_i) \).

The **excess demand** correspondence \( z : \mathbb{R}^m \rightarrow \mathbb{R}^m \) of a Fisher market \((u, b)\), which takes as input prices and outputs a set of excess demands at those prices, is defined as the difference between the

\(^8\)The definition of elasticity can be extended to non-unique and non-differentiable demands with additional care (see for instance Cheung [20]).
aggregate demand for and the supply of each good: i.e., \(z(p) = q(p) - 1_m\) where \(1_m\) is the vector of ones of size \(m\), and \(q(p) - 1_m = \{x - 1_m \mid \forall x \in q(p)\}\).

The discrete tâtonnement process for Fisher markets is a decentralized, natural price adjustment, defined as:

\[
p(t+1) = p(t) + G(g(t))
\]

for \(t = 0, 1, 2, \ldots\)

\[
g(t) \in z(p(t))
\]

\[
p(0) \in \mathbb{R}^m
\]

where \(G : \mathbb{R}^m \rightarrow \mathbb{R}^m\) is a coordinate-wise monotonic function such that, for all \(j \in [m], x, y \in \mathbb{R}^m\), if \(x_j \geq y_j\), then \(G_j(x) \geq G_j(y)\). Intuitively, tâtonnement is an auction-like process in which the sellers increase (resp. decrease) the prices of goods whenever the demand (resp. supply) is greater than the supply (resp. demand).

### 2.5 Homothetic Fisher Markets

Suppose that \((u, b)\) is a continuous, concave, and homogeneous Fisher market. The optimal solutions \((X^*, p^*)\) to the primal and dual of Eisenberg-Gale program constitute a competitive equilibrium of \((u, b)\) [31, 34, 45]:

**Primal**

\[
\max_{X \in \mathbb{R}^{n \times m}} \sum_{i \in [n]} b_i \log(u_i(x_i))
\]

subject to \(\sum_{i \in [n]} x_{ij} \leq 1 \quad \forall j \in [m]\)

**Dual**

\[
\min_{p \in \Delta_m} \sum_{j \in [m]} p_j + \sum_{i \in [n]} [b_i \log(u_i(p, b_i)) - b_i]
\]

Recently, Goktas et al. [40] proposed a convex program that is equivalent to the Eisenberg-Gale convex program, but whose optimal value differs from that of the Eisenberg-Gale convex program by an additive constant. The authors state their results only for continuous, concave, and homogeneous Fisher markets; however, their proof is valid for all homothetic Fisher markets, including those in which the buyers’ utility functions are non-concave:

**Theorem 2.2 (Goktas et al. [40]).** The optimal solutions \((X^*, p^*)\) to the following primal and dual convex programs correspond to competitive equilibrium allocations and prices, respectively, of the homothetic Fisher market \((u, b)\):

**Primal**

\[
\max_{X \in \mathbb{R}^{n \times m}} \sum_{i \in [n]} \left[ b_i \log u_i \left( \frac{x_i}{b_i} \right) + b_i \right]
\]

subject to \(\sum_{i \in [n]} x_{ij} \leq 1 \quad \forall j \in [m]\)

**Dual**

\[
\min_{p \in \Delta_m} \varphi(p) \doteq \sum_{j \in [m]} p_j - \sum_{i \in [n]} [b_i \log(e_i(p, 1))]
\]

Since the objective function of the primal in Theorem 2.2 is non-concave without assuming concavity of utilities \(u\), strong duality does not hold; however, the dual is still guaranteed to be convex [16]. This observation suggests that even if the problem of computing competitive equilibrium allocations is non-concave, the problem of computing competitive equilibrium prices can still be convex. Additionally, since this convex program differs from the Eisenberg-Gale program by an additive constant, we obtain as a corollary that solutions to the Eisenberg-Gale program also

---

We note the dual as presented here was formulated by Goktas et al. [40].
correspond to competitive equilibria in all homothetic Fisher markets, including those in which the buyers’ utility functions are non-concave.

An interesting property of this convex program is that its dual expresses competitive equilibrium prices via expenditure functions, and just like the Eisenberg-Gale program’s dual objective [22, 32], the gradient of its objective at any price is equal to the negative excess demand in the market at those prices. Define

\[
\varphi(p) = \sum_{j \in [m]} p_j - \sum_{i \in [n]} b_i \log \left( \partial v_i e_i(p, v_i) \right)
\]  

(6)

**Theorem 2.3 (Goktas et al. [40]).** Given any homothetic Fisher market \((u, b)\), the subdifferential of the dual of the program in Theorem 2.2 at any price \(p\) is equal to the negative excess demand in \((u, b)\) at price \(p\): i.e., \(\partial p \varphi(p) = -z(p)\).

Cheung et al. [22] define a class of markets called convex potential function (CPF) markets. A market is a CPF market, if there exists a convex potential function \(\varphi\) such that \(\partial p \varphi(p) = -z(p)\). A corollary of Theorem 2.3 is that all homothetic Fisher markets are CPF markets. This in turn implies that mirror descent on \(\varphi\) over the unit simplex is equivalent to tâtonnement in all homothetic Fisher markets, for some monotone function \(G\). Using this equivalence, we can pick a particular kernel function \(h\), and then potentially use Theorem 2.1 to establish convergence rates for tâtonnement.

Unfortunately, tâtonnement does not converge to equilibrium prices in all homothetic Fisher markets, e.g., linear Fisher markets [29], which suggests the need for additional restrictions on the class of homothetic Fisher markets. Goktas et al. [40] suggest the maximum absolute value of the Marshallian price demand elasticity, i.e., \(c = \max_{j,k,l} \max_{(p,b) \in \Delta_m \times \Delta_n} \left| \ell_{i,j}(p,b) \right|\) as a possible market parameter to use to establish a convergence rate of \(O((1+c)/\varepsilon)\). However, this suggestion is incorrect, as tâtonnement converges in Leontief Fisher markets, where \(c \to \infty\) in these markets. Previous results also suggest that it is unlikely that own-price Marshallian demand elasticity could be enough either, since the proof techniques used in work that makes this assumption require one to quantify the direction of the change in demand as a function of the change in the prices of the other goods, and hence only apply when one assumes WGS or WGC [27].

3 MARKET PARAMETERS

One of the main contributions of this paper is the observation that the maximum absolute value of the price elasticity of Hicksian demand in a homothetic Fisher market is sufficient to analyze the convergence of tâtonnement. To this end, in this section, we analyze Hicksian demand price elasticity, expose some of its properties in homothetic markets, and argue why it is a natural parameter to consider in the analysis of tâtonnement.

We first note that for Leontief utilities, the Hicksian cross-price elasticity of demand is equal to 0, while for linear utilities the Hicksian cross-price elasticity of demand is, by convention, \(\infty\). For Cobb-Douglas utilities, the Hicksian cross-price elasticity of demand is strictly positive and upper bounded by 1 (assuming valuations are normalized, i.e., \(\sum_{j \in [m]} a_{ij} = 1\), but it is not the same for all pairs of goods. Note that the behavior of the Hicksian cross-price elasticity of demand is radically different than that of the Marshallian price elasticity of demand, for which the elasticities of linear, Cobb-Douglas, and Leontief utilities are respectively given as \(\infty\), 0, and \(-\infty\). A taxonomy of utility classes as a function of price elasticity of demand (both Marshallian and Hicksian) is shown in Figure 1b.
We start our analysis with following lemma, which shows that the Hicksian price elasticity of demand is constant across all utility levels in homothetic Fisher markets. This property implies that the Hicksian demand price elasticity at one unit of utility provides sufficient information about the market’s reactivity to changes in prices, even without any information about the buyers’ utility levels. This information is crucial when trying to bound the changes in Hicksian demand from one iteration of tâtonnement to another, since buyers’ utilities can change.\footnote{We include all omitted results and proofs in Appendix A.}

**Lemma 3.1.** For any Hicksian demand $h_i$ associated with a homogeneous utility function $u_i$, for all $j, k \in [m], \ p \in \mathbb{R}^m_+, v_j \in \mathbb{R}_+$, it holds that $\epsilon_{h_i, p_k} (p, v_j) = \epsilon_{h_i, p_k} (p, 1) = 1$.

With the above lemma in hand, we now explain why the Hicksian demand price elasticity\footnote{Going forward we refer to the Hicksian price elasticity of demand, as simply Hicksian demand elasticity, because Hicksian price elasticity of demand w.r.t. utility level 1 is equal to 1.} is a better market parameter by which to analyze the convergence of tâtonnement than the Marshallian demand price elasticity. Cheung et al. [20, 22] use the dual of the Eisenberg-Gale program as a potential to measure the progress that tâtonnement makes at each step, for (nested) CES and Leontief utilities. Under these functional forms, the authors are able to explain a change in the value of the buyers’ indirect utilities as a function a change in prices, based on which they bound the change in the second term of the dual $\sum_{i \in [n]} b_i \log(u_i(p, b_i)) - b_i$ from one time period to the next. Using this bound, they show that tâtonnement makes steady progress towards equilibrium.

However, in general homothetic Fisher markets, knowing how much the Marshallian demand for each good changes from one iteration of tâtonnement to another does not tell us how much the buyers’ utilities change. More concretely, suppose that the Marshallian demand of a buyer $i$ has changed by an additive vector $\Delta d_i$ from time $t$ to time $t+1$, then the difference in indirect utilities from one period to another is given by $u_i(d_i^{(t+1)}) - u_i(d_i^{(t)}) = u_i(d_i^{(t)} + \Delta d_i) - u_i(d_i^{(t)})$. Without additional information about the utility functions, e.g., Lipschitz continuity, it is impossible to bound this difference, because utilities can change by an unbounded amount from one period to another. Hence, even if the Marshallian price elasticity of demand and the changes in prices from one period to another were known, it would only allow us to bound the difference in demands, and not the difference in indirect utilities. To get around this difficulty, one could consider making an assumption about the boundedness of the indirect utility function’s price elasticity, or the utility function’s Lipschitz-continuity, but such assumptions would be economically unjustified, since utility functions are merely representations of preference orderings without any inherent meaning of their own.

We can circumvent this issue by instead looking at the dual of the convex program in Theorem 2.2. In this dual, the indirect utility term is replaced by the expenditure function. The advantage of this formulation is that if one knows the amount by which prices change from one iteration to the next, as well as the Hicksian demand elasticity, then we can easily bound the change in spending from one period to another.

The following lemma is crucial to proving the convergence of tâtonnement. This lemma allows us to bound the changes in buyer spending across all time periods, thereby allowing us to obtain a global convergence rate. In particular, it shows that the change in spending between consecutive iterations of tâtonnement can be bounded as a function of the prices and the Hicksian demand elasticity.

More formally, suppose that we would like to bound the percentage change in expenditure at one unit of utility from one iteration to another, i.e., $e_i(p^{(t)} + \Delta p, 1)$, using a first order Taylor expansion of $e_i(p^{(t)} + \Delta p, 1)$ around $p^{(t)}$. By Taylor’s theorem [42], we have: $e_i(p^{(t)} + \Delta p, 1) = e_i(p^{(t)}, 1) +$
We can now apply Shephard’s lemma \[67\], a corollary of the envelope theorem \[1, 58\], to the

\[\langle \nabla_p e_i(p^{(t)}, 1), \Delta p \rangle + \frac{1}{2} \left\langle \nabla_p^2 e_i(p^{(t)} + c\Delta p, 1) \Delta p, \Delta p \right\rangle\]

for some \( c \in (0, 1) \). Re-organizing terms around, we get

\[e_i(p^{(t+1)}, 1) - e_i(p^{(t)}, 1) = \langle \nabla_p e_i(p^{(t)}, 1), \Delta p \rangle + \frac{1}{2} \left\langle \nabla_p^2 e_i(p^{(t)} + c\Delta p, 1) \Delta p, \Delta p \right\rangle.\]

Dividing both sides by \( e_i(p^{(t)}, 1) \) we obtain:

\[
\frac{\langle \nabla_p e_i(p^{(t)}, 1), \Delta p \rangle + \frac{1}{2} \left\langle \nabla_p^2 e_i(p^{(t)} + c\Delta p, 1) \Delta p, \Delta p \right\rangle}{e_i(p^{(t)}, 1)}
\]

We can now apply Shephard’s lemma \[67\], a corollary of the envelope theorem \[1, 58\], to the numerators, which allows us to conclude that for all buyers \( i \in [n] \), \( \nabla_p e_i(p, v_i) = h_i(p, v_i) \). Next, using the definition of the expenditure function in the denominator, we obtain the following:

\[
= \frac{\langle h_i(p^{(t)}, 1), \Delta p \rangle}{\langle h_i(p^{(t)}, 1), p^{(t)} \rangle} + \frac{1}{2} \left\langle \frac{\nabla_p^2 e_i(p^{(t)} + c\Delta p, 1) \Delta p, \Delta p}{e_i(p^{(t)}, 1)} \right\rangle
\]

Note that since the Hicksian elasticity of demand is bounded, \( h_i(p^{(t)}, 1) \) is unique and the above expression is well-defined. If the change in prices is bounded, and the Hicksian demand elasticity is known, then one can bound the first term in Equation (7) with ease. It remains to be seen if the second term can be bounded. The following lemma provides an affirmative answer to that question. In particular, we show that the second-order error term in the Taylor approximation above can be bounded as a function of the maximum absolute value of the Hicksian demand elasticity. We note that in the following lemma, by Lemma A.9, the Marshallian demand is unique, because the Hicksian demand is a singleton for bounded elasticity of Hicksian demand.

**Lemma 3.2.** Fix \( i \in [n] \) and \( t \in \mathbb{N}_+ \) and let \( \Delta p = p^{(t+1)} - p^{(t)} \). Suppose that \( |\Delta p_j| \leq \frac{1}{4} \), then for all buyers \( i \in [n] \), and for some \( c \in (0, 1) \), it holds that:

\[
\left| \frac{\frac{\nabla^2 e_i(p^{(t)} + c\Delta p, 1) \Delta p, \Delta p}{2}}{\langle h_i(p^{(t)}, 1) \rangle} \right| \leq \frac{5\epsilon}{6} \sum_j \frac{(\Delta p_j)^2}{p_j^{(t)}} \frac{1}{2} \frac{\partial_i h_j(p^{(t)} + c\Delta p, b_i)}{d_{ij}(p^{(t)} + c\Delta p, b_i)}
\]

where \( \epsilon = \max_{p \in \mathbb{A}_{m,j,k}} |e_{ij,p,k}(p, 1)| \).

**Proof of Lemma 3.2.** By Shephard’s lemma \[67\] (Lemma A.1, Appendix A), it holds that

\[
\frac{b_i \left\langle \nabla_p^2 e_i(p^{(t)} + c\Delta p, 1) \Delta p, \Delta p \right\rangle}{2 \langle h_i(p^{(t)}, 1) \rangle} = \frac{b_i \left\langle \nabla_p h_i(p^{(t)} + c\Delta p, 1) \Delta p, \Delta p \right\rangle}{2 \langle h_i(p^{(t)}, 1) \rangle}
\]

(Shephard’s Lemma)

\[
\leq \frac{b_i \sum_{j,k} |\Delta p_j| |\partial_{p_k} h_{ij}(p^{(t)} + c\Delta p, 1)| |\Delta p_k|}{2 \langle h_i(p^{(t)}, 1) \rangle}
\]

(10)

\[
= \frac{b_i \sum_{j,k} |\Delta p_j| \sqrt{|\partial_{p_k} h_{ij}(p^{(t)} + c\Delta p, 1)| |\partial_{p_k} h_{ij}(p^{(t)} + c\Delta p, 1)| |\Delta p_k|}}{2 \langle h_i(p^{(t)}, 1) \rangle}
\]

(12)
where the last was obtained from the symmetry of $\nabla^2 \epsilon_i(p, v_i) = \nabla^2 \epsilon_i(p, v_i)^T$ for all $i \in [n], p \in \mathbb{R}_+^m, v_j \in \mathbb{R}_+ [55]$, which combined with Shepherd’s lemma gives us $\nabla_p h_i(p, v_i) = \nabla_p h_i(p, v_i)^T$, i.e., for all $j, k \in [m], \partial_{p_j} h_{ij}(p, v_i) = \partial_{p_k} h_{ij}(p, v_i)$.

Define the Hicksian demand elasticity of buyer $i$ for good $j$ w.r.t. the price of good $k$ as $\epsilon_{h_{ij},pk}(p, v_i) = \partial_{p_k} h_{ij}(p, v_i) \frac{p_k}{h_{ij}(p, v_i)}$. Since utility functions are homogeneous, by Lemma 3.1 we have for all $v_i \in \mathbb{R}_+, \epsilon_{h_{ij},pk}(p, v_i) = \partial_{p_k} h_{ij}(p, v_i) \frac{p_k}{h_{ij}(p, v_i)} = \partial_{p_k} h_{ij}(p, 1) \frac{p_k}{h_{ij}(p, 1)}$. Re-organizing expressions, we get $\partial_{p_k} h_{ij}(p, 1) = \epsilon_{h_{ij},pk}(p, 1) \frac{h_{ij}(p, 1)}{p_k}$. Going back to Equation (13), we get:

\[
\frac{\sum_{j,k} |\Delta p_j|}{2} \sqrt{|\partial_{p_j} h_{ik}(p^{(i)} + c\Delta p, 1)|} \sqrt{|\partial_{p_k} h_{ij}(p^{(i)} + c\Delta p, 1)|} |\Delta p_k| \langle h_i, p^{(i)} \rangle
\]

Letting $\epsilon = \max_{p \in \mathbb{R}_+^m, v_i \in \mathbb{R}_+, j,k \in [m]} |\epsilon_{h_{ij},pk}(p, v_i)|$. Note that since utility functions are homogeneous, by Lemma 3.1 we have $\epsilon = \max_{p \in \mathbb{R}_+^m, v_i \in \mathbb{R}_+, j,k \in [m]} |\epsilon_{h_{ij},pk}(p, v_i)| = \max_{p \in \mathbb{R}_+^m, j,k \in [m]} |\epsilon_{h_{ij},pk}(p, 1)|$, which gives us:

\[
\frac{\epsilon b_1}{2} \sum_{j,k} |\Delta p_j| \sqrt{\frac{h_{ik}(p^{(i)} + c\Delta p, 1)}{\frac{1}{p_j} p^{(i)} + c\Delta p}} \sqrt{\frac{h_{ij}(p^{(i)} + c\Delta p, 1)}{\frac{1}{p_k} p^{(i)} + c\Delta p}} |\Delta p_k| \langle h_i(p^{(i)}, 1), p^{(i)} \rangle
\]

Since for all $j \in [m], \frac{|\Delta p_j|}{p_j} \leq \frac{1}{4}$, we have that for all $j \in [m]$ and for all $c \in [0, 1], p^{(i)} + c\Delta p_j \geq \frac{3}{4} p^{(i)}$, which gives:

\[
\frac{\epsilon b_1}{2} \sum_{j,k} |\Delta p_j| \sqrt{\frac{h_{ik}(p^{(i)} + c\Delta p, 1)}{\frac{1}{p_j} p^{(i)} + c\Delta p}} \sqrt{\frac{h_{ij}(p^{(i)} + c\Delta p, 1)}{\frac{1}{p_k} p^{(i)} + c\Delta p}} |\Delta p_k| \langle h_i(p^{(i)}, 1), p^{(i)} \rangle
\]

\[
= \frac{2\epsilon b_1}{3} \sum_{j,k} |\Delta p_j| \sqrt{\frac{h_{ij}(p^{(i)} + c\Delta p, 1)}{\frac{1}{p_j} p^{(i)} + c\Delta p}} \sqrt{\frac{h_{ik}(p^{(i)} + c\Delta p, 1)}{\frac{1}{p_k} p^{(i)} + c\Delta p}} |\Delta p_k| \langle h_i(p^{(i)}, 1), p^{(i)} \rangle
\]

\[
= \frac{2\epsilon b_1}{3} \sum_{j,k \in [m]} \sqrt{\frac{|\Delta p_j|^2}{p_j}} h_{ij}(p^{(i)} + c\Delta p, 1) h_{ik}(p^{(i)} + c\Delta p, 1) \frac{|\Delta p_k|^2}{p_k} \langle h_i(p^{(i)}, 1), p^{(i)} \rangle
\]
Applying the AM-GM inequality, i.e., for all $x, y \in \mathbb{R}_+$, $\frac{x^t y^t}{2} \geq \sqrt{xy}$, to the sum inside the numerator above, we obtain:

$$
\leq 2\epsilon b_i \sum_{j,k \in [m]} \frac{1}{2} \left( \frac{|\Delta p_j|^2}{p_j^{(t)}} h_{ij}(p^{(t)} + c\Delta p, 1) + h_{ik}(p^{(t)} + c\Delta p, 1) \frac{|\Delta p_k|^2}{p_k^{(t)}} \right) \langle h_i(p^{(t)}, 1), p^{(t)} \rangle
$$

$$
\leq 2\epsilon b_i \sum_{j} \frac{1}{3} \frac{(|\Delta p_j|^2)^2}{p_j^{(t)}} h_{ij}(p^{(t)} + c\Delta p, 1) \langle h_i(p^{(t)}, 1), p^{(t)} \rangle
$$

Since for all $j \in [m]$, $\frac{|\Delta p_j|^2}{p_j^{(t)}} \leq \frac{1}{4}$, we have for all $c \in [0, 1]$ that $\frac{1}{2} \sum_{j} h_{ij}(p^{(t)}, 1)(p_j^{(t)} + c\Delta p_j) \leq \sum_{j} h_{ij}(p^{(t)}, 1)p_j^{(t)}$:

$$
\leq 2\epsilon b_i \frac{1}{3} \sum_{j} \frac{(|\Delta p_j|^2)^2}{p_j^{(t)}} h_{ij}(p^{(t)} + c\Delta p, 1)
$$

$$
= \frac{5\epsilon b_i}{6} \sum_{j} \frac{(|\Delta p_j|^2)^2}{p_j^{(t)}} h_{ij}(p^{(t)} + c\Delta p, 1)
$$

$$
\leq \frac{5\epsilon b_i}{6} \sum_{j} h_{ij}(p^{(t)} + c\Delta p, 1)(p_j^{(t)} + c\Delta p_j)
$$

$$
= \frac{5\epsilon}{6} \sum_{j} \frac{(|\Delta p_j|^2)^2}{p_j^{(t)}} d_{ij}(p^{(t)} + c\Delta p, b_i)
$$

(Corollary A.7, Appendix A)

(Lemma A.9, Appendix A)

Because we can bound the change in the expenditure function from one iteration of tâtonnement to the next, the Hicksian price elasticity of demand is a better tool with which to analyze the convergence of tâtonnement than Marshallian price elasticity of demand. Additionally, as shown previously by Cheung et al. [22] (Lemma A.3, Appendix A), we can further upper bound the price terms in Lemma 3.2 by the KL divergence between the two prices. In light of Theorem 2.1, this result suggests that running mirror descent with KL divergence as the Bregman divergence on the dual of the convex program in Theorem 2.2 could result in a tâtonnement update rule that converges to a competitive equilibrium.

4 CONVERGENCE BOUNDS FOR ENTROPIC TÂTONNEMENT

Notation. In this section, we adopt the following conventions for notational convenience: At time $t$ in the tâtonnement process, we denote buyer $i$’s Marshallian demand for good $j$ i.e., $d_{ij}(p^{(t)}, b_i)$, by $d_{ij}^{(t)}$; the aggregate demand for good $j$, i.e., for all goods $j \in [m]$, $q_j(p^{(t)})$, by $q_j^{(t)}$; and the buyer $i$’s Hicksian demand for good $j$ at utility level 1, i.e., $h \in h_{ij}(p^{(t)}, 1)$, by $h_{ij}^{(t)}$.

In this section, we analyze the rate of convergence of entropic tâtonnement, which corresponds to the tâtonnement process given by mirror descent with weighted entropy as the kernel function, i.e., entropic descent. This particular update rule reduces to Equations (4) to (5), and has been the focus of previous work [22].
We provide a sketch of the proof used to obtain our convergence rate in this section. The omitted lemmas and proofs can be found in Appendix A.

At a high level, our proof follows Cheung et al.’s proof technique for Leontief Fisher markets [22], although we encounter different technical challenges inherent to the abstraction cause by homothetic markets at a lower level. This proof technique works as follows. First, we prove that under certain assumptions, the condition required by Theorem 2.1 holds when $f$ is the convex potential function for homothetic Fisher markets as defined in Equation (6), i.e., $f = \varphi$. For these assumptions to be valid, we need to set $\gamma$ to be greater than a quadratic function of the maximum absolute value of the price elasticity of the Hicksian demand and the maximum Marshallian demand, for all goods throughout the tâtonnement process. Further, since $\gamma$ needs to be set at the outset, we need to upper bound $\gamma$. To do so, we derive a bound on the maximum demand for any good during tâtonnement in all homothetic Fisher markets, which in turn allows us to derive an upper bound on $\gamma$. Finally, we use Theorem 2.1 to obtain the convergence rate of $O(1 + \varepsilon/\varepsilon')$.

The following lemma derives the conditions under which the antecedent of Theorem 2.1 holds for entropic tâtonnement.

**Lemma 4.1.** Consider a homothetic Fisher market $(u, b)$ and let $\varepsilon = \max_{p \in \Delta, i, j \in [m]} |\varepsilon_{h, j}, p_i (p, 1)|$. Then, the following holds for entropic tâtonnement when run on a homothetic Fisher market $(u, b)$: for all $t \in \mathbb{N}$,

$$\varphi(p^{(t+1)}) - \ell_\varphi(p^{(t+1)}, p^{(t)}) \leq \varphi(p^{(t+1)}) - \varphi(p^{(t)}) + z(p^{(t)}) \cdot \left(p^{(t+1)} - p^{(t)}\right)$$

where $\gamma = \max_{j \in [m], t \in \mathbb{N}, e \in [0,1]} \{1 + q_j(p^{(t)} + c\Delta p, b)\} \left(6 + \frac{85\varepsilon}{12} + \frac{25\varepsilon^2}{72}\right)$.

**Proof of Lemma 4.1.**

$$\varphi(p^{(t+1)}) - \ell_\varphi(p^{(t+1)}, p^{(t)})$$

$$= \varphi(p^{(t+1)}) - \varphi(p^{(t)}) + z(p^{(t)}) \cdot \left(p^{(t+1)} - p^{(t)}\right)$$

$$= \sum_{j \in [m]} \left(p_j^{(t)} + \Delta p_j\right) - \sum_{i \in [n]} b_i \log \left(e_i(p^{(t+1)}, 1)\right) - \sum_{j \in [m]} p_j^{(t)} + \sum_{i \in [n]} b_i \log \left(e_i(p^{(t)}, 1)\right) + \sum_{j \in [m]} z_j(p^{(t)}) \Delta p_j$$

$$= \sum_{j \in [m]} \left(p_j^{(t)} + \Delta p_j\right) - \sum_{i \in [n]} b_i \log \left(e_i(p^{(t+1)}, 1)\right) - \sum_{j \in [m]} p_j^{(t)} + \sum_{i \in [n]} b_i \log \left(e_i(p^{(t)}, 1)\right) + \sum_{j \in [m]} (q_j^{(t)} - 1) \Delta p_j$$

$$= \sum_{j \in [m]} \Delta p_j q_j^{(t)} - \sum_{i \in [n]} b_i \log \left(e_i(p^{(t+1)}, 1)\right) + \sum_{i \in [n]} b_i \log \left(e_i(p^{(t)}, 1)\right)$$

$$= \left(\Delta p, q^{(t)}\right) + \sum_{i \in [n]} b_i \log \left(\frac{e_i(p^{(t)}, 1)}{e_i(p^{(t+1)}, 1)}\right)$$

$$= \left(\Delta p, q^{(t)}\right) + \sum_{i \in [n]} b_i \log \left(\frac{e_i(p^{(t)}, 1)}{e_i(p^{(t)}, 1) + e_i(p^{(t+1)}, 1) - e_i(p^{(t)}, 1)}\right)$$

$$= \left(\Delta p, q^{(t)}\right) + \sum_{i \in [n]} b_i \log \left(1 - \frac{e_i(p^{(t+1)}, 1) - e_i(p^{(t)}, 1)}{e_i(p^{(t)}, 1)} \left(1 + \frac{e_i(p^{(t+1)}, 1) - e_i(p^{(t)}, 1)}{e_i(p^{(t)}, 1)}\right)^{-1}\right)$$

where the last line is obtained by simply noting that $\forall a, b \in \mathbb{R}, \frac{a}{a+b} = 1 - \frac{b}{a} (1 + \frac{b}{a})^{-1}$. Using Lemma A.12, we then obtain:

$$\varphi(p^{(t+1)}) - \ell_\varphi(p^{(t+1)}, p^{(t)})$$
\[
\Delta \mathbf{q}^{(t)} = \sum_{i \in [m]} \left( \frac{4}{3} + \frac{20 \epsilon}{27} \right) \sum_{l \in [m]} \frac{q_i^{(t)}}{p_l^{(t)}} (\Delta p_l)^2 + \left( \frac{5 \epsilon^2}{6} + \frac{25 \epsilon^2}{324} \right) \sum_{l \in [m]} \frac{q_l(p_l^{(t)} + c\Delta \mathbf{p}, b_l)^2}{p_l^{(t)}} (\Delta p_l)^2 - \left( \frac{d_i^{(t)}}{\Delta \mathbf{p}} \right)
\]

where the last line follows from Lemma A.3. Continuing,
\[
\leq \frac{4}{3} + \frac{20 \epsilon}{27} \sum_{l \in [m]} \frac{q_l^{(t)}}{p_l^{(t)}} (\Delta p_l)^2 + \left( \frac{5 \epsilon^2}{6} + \frac{25 \epsilon^2}{324} \right) \sum_{l \in [m]} \frac{q_l(p_l^{(t)} + c\Delta \mathbf{p}, b_l)^2}{p_l^{(t)}} (\Delta p_l)^2
\]

Combining Lemma 4.1 with Theorem 2.1, we obtain our main result, namely a worst-case convergence rate of \(O((1+\epsilon^2)t)\) for entropic tâtonnement in homothetic Fisher markets.

**Theorem 4.2.** Suppose \((\mathbf{u}, \mathbf{b})\) is homothetic Fisher market and \(\epsilon = \max_{\mathbf{p} \in \Delta m, j, k \in [m]} |\epsilon_{h_j, p_k}(\mathbf{p}, \mathbf{1})|\). Then, the following holds for entropic tâtonnement for all \(t \in \mathbb{N}:
\]
\[
\varphi(p^{(t)}) - \varphi(p^*) \leq \frac{\gamma_{\text{KL}}(\mathbf{p}^*, \mathbf{p}^0)}{t},
\]

where \(\gamma = \max_{j \in [m], t \in \mathbb{N}, c \in [0,1]} \{q_j(p^{(t)} + c\Delta \mathbf{b}, b_j)\} \left(6 + \frac{85 \epsilon}{12} + \frac{25 \epsilon^2}{72}\right) \delta_{\text{KL}}(p^{(t)} + \Delta \mathbf{p}, \mathbf{p}^{(t)})\).
Lemma 4.3 (Step size for homothetic complements Markets). If tâtonnement is run on a complements Fisher market \((u, b)\), then, for all \(t \in \mathbb{N}_+\), prices satisfy \(\left| \frac{\Delta p_j}{p_j} \right| \leq \frac{1}{4}\) if and only if
\[
y \geq \max \left\{ a_j^{(0)}, \max_{\mathcal{P} \in \mathcal{P}_m} \max_{k \in [m]} \left\{ \frac{h_{ij}(p, 1)}{h_{ik}(p, 1)} \right\} \right\}.
\]

5 CONCLUSION

We identified the maximum absolute value of the Hicksian price elasticity of demand as a sufficient parameter by which to analyze convergent and non-convergent behavior of tâtonnement in homothetic Fisher markets. We then showed that together with the KL divergence associated with a change in prices, we can use it to bound the percentage change in the expenditure of one unit of utility, assuming bounded price changes. This observation motivated us to consider analyzing the convergence of mirror descent with KL divergence on a recently proposed \([40]\) convex potential, making use of the expenditure function to characterize competitive equilibrium prices in homothetic Fisher markets. An important property of this convex potential is that its gradient is equal to the negative excess demand in the market, implying that mirror descent on it is equivalent to tâtonnement, an observation used to prove previous convergence results regarding tâtonnement in Fisher markets \([22]\). Using the bound we derived on the change in the expenditure function as a function of the change in prices, we then showed that the potential function we considered is Bregman-smooth w.r.t. the KL divergence throughout a trajectory of tâtonnement. Combining this result with the sublinear convergence rate of mirror descent for Bregman-smooth functions \([13]\), we concluded that tâtonnement converges at a rate of \(O((1+\varepsilon^2)/T)\), where \(\varepsilon\) is the maximum absolute value of the price elasticity of Hicksian demand across all buyers. Our result not only generalizes existing convergence results for CES and nested CES Fisher markets, but extends them beyond Fisher markets with concave utility functions. Our convergence rate covers the full spectrum of (nested) CES utilities, including Leontief and linear utilities, unifying previously existing disparate convergence and non-convergence results. In particular, for \(\varepsilon = 0\), i.e., Leontief markets, we recover the best-known convergence rate of \(O(1/T)\), and as \(\varepsilon \to \infty\), i.e., linear Fisher markets, we obtain the non-convergent behavior of tâtonnement.

Future work could investigate the space of homogeneous utility functions with negative cross-price elasticity of Hicksian demand to possibly derive faster convergence rates than those provided in this paper. Additionally, it remains to be seen if the bound we have provided in this paper is tight; the greatest lower bound known for the convergence of tâtonnement in homothetic Fisher markets is \(O(1/T)\) for Leontief markets \([23]\), leaving space for improvement. Finally, Lemma 3.1 suggests that to extend convergence results for tâtonnement beyond homothetic domains, one might have to consider the Hicksian demand elasticity w.r.t. utility levels—rather than price.

REFERENCES


A

OMMUTED RESULTS AND PROOFS

We start by presenting the first lemma, which shows that the utility level elasticity of Hicksian demand is equal to 1 in homothetic Fisher markets.

\textbf{Lemma 3.1.} For any Hicksian demand $\mathbf{h}_i$ associated with a homogeneous utility function $u_i$, for all $j,k \in [m]$, $\mathbf{p} \in \mathbb{R}^m_+$, $v_i \in \mathbb{R}_+$, it holds that $\varepsilon_{h_{ij},p_k}(\mathbf{p}, v_i) = \varepsilon_{h_{ij},p_k}(\mathbf{p}, 1) = 1$.

\textbf{Proof of Lemma 3.1.} Recall from Goktas et al. \cite{Goktas2010} that for homogeneous utility functions, the Hicksian demand is homogeneous in $v$, i.e., for all $\lambda \geq 0$, $\mathbf{h}_i(\mathbf{p}, \lambda v) = \lambda \mathbf{h}_i(\mathbf{p}, v)$. Hence, we have:

\begin{align}
\varepsilon_{h_{ij},p_k}(\mathbf{p}, v_i) &= \partial_{p_k} h_{ij}(\mathbf{p}, v_i) \frac{p_k}{h_{ij}(\mathbf{p}, v_i)} \\
&= v_i \partial_{p_k} h_{ij}(\mathbf{p}, 1) \frac{p_k}{v_i h_{ij}(\mathbf{p}, 1)} \\
&= \frac{p_k}{h_{ij}(\mathbf{p}, 1)} \partial_{p_k} h_{ij}(\mathbf{p}, 1) \\
&= \varepsilon_{h_{ij},p_k}(\mathbf{p}, 1) \\
&= \varepsilon_{h_{ij},p_k}(\mathbf{p}, 1) = 1
\end{align}

(15) (16) (17) (18) (19)

Additionally, looking back at Equation (18), since Hicksian demand is homogeneous of degree 1 for homogeneous utility function (see Lemma 5 of Goktas et al. \cite{Goktas2010}), by Euler’s theorem for homogeneous functions \cite{Euler1755}, we have:

$$\frac{p_k}{h_{ij}(\mathbf{p}, 1)} \partial_{p_k} h_{ij}(\mathbf{p}, 1) = \frac{h_{ij}(\mathbf{p}, 1)}{h_{ij}(\mathbf{p}, 1)} = 1.$$  

(20)

We recall Shephard’s lemma which was used in the Equation (7):

\textbf{Lemma A.1 (Shephard’s Lemma \cite{Shephard1939, Shephard1960, Shephard1976}).} Let $e_i(\mathbf{p}, v_i)$ be the expenditure function of buyer $i$ and $\mathbf{h}_i(\mathbf{p}, v_i)$ be the Hicksian demand set of buyer $i$. The subdifferential $\partial_{\mathbf{p}} e_i(\mathbf{p}, v_i)$ is the Hicksian demand at prices $\mathbf{p}$ and utility level $v_i$, i.e., $\partial_{\mathbf{p}} e_i(\mathbf{p}, v_i) = \mathbf{h}_i(\mathbf{p}, v_i)$. 

(21)
We first prove that by setting $\gamma$ to be 5 times the maximum demand for any good throughout the entropic tâtonnement process, we can bound the change in the prices of goods in each round. We will use the fact that the change in the price of each good is bounded as an assumption in most of the following results.

**Lemma A.2.** Suppose that entropic tâtonnement process is run for all $t \in [T] \subseteq \mathbb{N}_+$ with $\gamma \geq 5 \max \{1, q_j^{(t)}\}$ and let $\Delta \mathbf{p} = \mathbf{p}^{(t+1)} - \mathbf{p}^{(t)}$. then the following holds for all $t \in \mathbb{N}$:

$$e^{-\frac{1}{2}} p_j^{(t)} \leq p_j^{(t+1)} \leq e^{\frac{1}{2}} p_j^{(t)} \text{ and } \frac{|\Delta p_j|}{p_j^{(t)}} \leq \frac{1}{4}$$

**Lemma A.2.** The price of a good $j \in [m]$ can at most increase by a factor of $e^{\frac{1}{2}}$:

$$p_j^{(t+1)} = p_j^{(t)} \exp \left\{ \frac{z_j(\mathbf{p}^{(t)})}{\gamma} \right\} = p_j^{(t)} \exp \left\{ \frac{q_j^{(t)} - 1}{\gamma} \right\} \leq p_j^{(t)} \exp \left\{ \frac{q_j^{(t)}}{5 \max \{1, q_j^{(t)}\}} \right\} \leq p_j^{(t)} e^{\frac{1}{2}}$$

and decrease by a factor of $e^{-\frac{1}{2}}$:

$$p_j^{(t+1)} = p_j^{(t)} \exp \left\{ \frac{z_j(\mathbf{p}^{(t)})}{\gamma} \right\} = p_j^{(t)} \exp \left\{ \frac{q_j^{(t)} - 1}{\gamma} \right\} \geq p_j^{(t)} \exp \left\{ \frac{-1}{\gamma} \right\} \geq p_j^{(t)} \exp \left\{ \frac{-1}{5 \max \{1, q_j^{(t)}\}} \right\} \geq p_j^{(t)} e^{-\frac{1}{2}}$$

Hence, we have $e^{-\frac{1}{2}} p_j^{(t)} \leq p_j^{(t+1)} \leq e^{\frac{1}{2}} p_j^{(t)}$. Subtracting $p_j^{(t)}$ from both sides and dividing by $p_j^{(t)}$, we obtain:

$$\frac{|\Delta p_j|}{p_j^{(t)}} = \frac{|p_j^{(t+1)} - p_j^{(t)}|}{p_j^{(t)}} \leq e^{\frac{1}{2}} - 1 \leq \frac{1}{4}$$

The following two results are due to Cheung et al. [22]. We include their proofs for completeness. They allow us to relate the change in prices to the KL-divergence.

**Lemma A.3 (Cheung et al. [22]).** Fix $t \in \mathbb{N}_+$ and let $\Delta \mathbf{p} = \mathbf{p}^{(t+1)} - \mathbf{p}^{(t)}$. Suppose that for all $j \in [m]$, $\frac{|\Delta p_j|}{p_j^{(t)}} \leq \frac{1}{4}$, then:

$$\frac{(\Delta p_j)^2}{p_j^{(t)}} \leq \frac{9}{2} \delta_{\text{KL}}(p_j^{(t)} + \Delta p_j, p_j^{(t)})$$

**Lemma A.3.** The bound $\log(x) \geq x - x^2$ for $|x| \leq \frac{1}{4}$ is used below:

$$\delta_{\text{KL}}(p_j^{(t)} + \Delta p_j, p_j^{(t)}) = (p_j^{(t)} + \Delta p_j)(\log(p_j^{(t)} + \Delta p_j)) - (p_j^{(t)}) + \Delta p_j - p_j^{(t)} \log(p_j) + p_j^{(t)} - \log(p_j)\Delta p_j$$

$$= -\Delta p_j + (p_j^{(t)} + \Delta p_j) \log \left( 1 + \frac{\Delta p_j}{p_j^{(t)}} \right)$$
\[
\begin{align*}
&\geq -\Delta p_j + (p_j^{(t)} + \Delta p_j) \left( \frac{\Delta p_j}{p_j^{(t)}} - \frac{11}{18} \left( \frac{\Delta p_j}{p_j^{(t)}} \right)^2 \right) \\
&\geq \frac{7}{18} \left( \frac{\Delta p_j}{p_j^{(t)}} \right)^2 \left( 1 - \frac{11}{7} \frac{\Delta p_j}{p_j^{(t)}} \right) \\
&= \frac{7}{18} \frac{17}{28} \left( \frac{\Delta p_j}{p_j^{(t)}} \right)^2 \\
&\geq \frac{2}{9} \left( \frac{\Delta p_j}{p_j^{(t)}} \right)^2
\end{align*}
\]

\[\Box\]

**Lemma A.4.** Fix \( t \in \mathbb{N}_+ \) and let \( \Delta p = p^{(t+1)} - p^{(t)} \). Suppose that \( \frac{|\Delta p_j|}{p_j} \leq \frac{1}{4} \), then for any \( c \in (0, 1) \), and \( A \in \mathbb{R}^{n \times m} \), and for all \( j \in [m] \):

\[
\frac{1}{b_i} \sum_{j \in [m]} \sum_{k \in [m]} a_{il} d_{ik}(p^{(t)} + c\Delta p, b_i) |\Delta p_j||\Delta p_k| \leq \frac{4}{3} \sum_{l \in [m]} \frac{a_{il}}{p_l^{(t)}} (\Delta p_l)^2
\]

**Lemma A.4.** First, note that since by our assumption the utilities are locally non-satiated, Walras’ law is satisfied, i.e., we have \( b_i = \sum_{k \in [m]} d_{ik}(p^{(t)} + c\Delta p, b_i)(p_k^{(t)} + c\Delta p_k) \);

\[
b_i \sum_{l \in [m]} \frac{a_{il}}{p_l^{(t)}} (\Delta p_l)^2 = \sum_{l \in [m]} \left( \sum_{k \in [m]} d_{ik}(p^{(t)} + c\Delta p, b_i)(p_k^{(t)} + c\Delta p_k) \right) \frac{d_{il}(p^{(t)} + c\Delta p, b_i)}{p_l^{(t)}} (\Delta p_l)^2 \]

\[
\geq \sum_{l \in [m]} \frac{\left( \sum_{k \in [m]} d_{ik}(p^{(t)} + c\Delta p, b_i)(p_k^{(t)} + c\Delta p_k) \right) a_{il}}{p_l^{(t)}} (\Delta p_l)^2 \]

\[
= \sum_{l \in [m]} \frac{\left( \sum_{k \in [m]} d_{ik}(p^{(t)} + c\Delta p, b_i)(\frac{3}{2}p_k^{(t)}) \right) a_{il}}{p_l^{(t)}} (\Delta p_l)^2 \]

\[
= \frac{3}{4} \sum_{l \in [m]} \sum_{k \in [m]} a_{il} d_{il}(p^{(t)} + c\Delta p, b_i) \frac{p_k^{(t)}}{p_l^{(t)}} (\Delta p_l)^2 \]

\[
= \frac{3}{4} \left[ \sum_{l \in [m]} a_{il} d_{il}(p^{(t)} + c\Delta p, b_i)(\Delta p_l)^2 + \sum_{l \in [m]} \sum_{k \neq l} a_{il} d_{ik}(p^{(t)} + c\Delta p, b_i) p_k^{(t)} \frac{p_l^{(t)}}{p_l^{(t)}} (\Delta p_l)^2 \right] \]

\[
= \frac{3}{4} \left[ \sum_{l \in [m]} a_{il} d_{il}(p^{(t)} + c\Delta p, b_i)(\Delta p_l)^2 + \sum_{k \in [m]} \sum_{k \neq l} a_{il} d_{ik}(p^{(t)} + c\Delta p, b_i) \left( \frac{p_l^{(t)}}{p_l^{(t)}} |\Delta p_l|^2 + \frac{p_l^{(t)}}{p_k^{(t)}} |\Delta p_k|^2 \right) \right] \]

Now, we apply the AM-GM inequality, i.e., for all \( x, y \in \mathbb{R}_+ \) since \( \sqrt{xy} \leq \frac{x+y}{2} \), we have:

\[
b_i \sum_{l \in [m]} \frac{a_{il}}{p_l^{(t)}} (\Delta p_l)^2 \geq \frac{3}{4} \sum_{l \in [m]} a_{il} d_{il}(p^{(t)} + c\Delta p, b_i)(\Delta p_l)^2 + \sum_{k < l} a_{il} d_{ik}(p^{(t)} + c\Delta p, b_i) \left( 2|\Delta p_l||\Delta p_k| \right) \]
\[\sum_{i \in [m]} \sum_{k \in [m]} a_{il} d_{ik} (p^{(t)} + c\Delta p, b_i) |\Delta p_j| |\Delta p_k| = \frac{3}{4} \sum_{i \in [m]} \sum_{k \in [m]} a_{il} d_{ik} (p^{(t)} + c\Delta p, b_i) |\Delta p_j| |\Delta p_k| \]

\[\sum_{i \in [m]} \sum_{k \in [m]} d_{ij}^{(t)} d_{ik}^{(t)} |\Delta p_j| |\Delta p_k| \leq \sum_{l \in [m]} \frac{d_{il}^{(t)}}{p_l^{(t)}} (\Delta p_l)^2\]

**Lemma A.5.** [22] For all \( j \in [m] \):

\[\frac{1}{b_j} \sum_{i \in [m]} \sum_{k \in [m]} d_{ij}^{(t)} d_{ik}^{(t)} |\Delta p_j| |\Delta p_k| \leq \sum_{l \in [m]} \frac{d_{il}^{(t)}}{p_l^{(t)}} (\Delta p_l)^2\]

**Lemma A.5.** First, note that by Walras’ law we have \( b_i = \sum_{k \in [m]} d_{ik}^{(t)} p_k^{(t)} \):

\[b_i \sum_{l \in [m]} \frac{d_{il}^{(t)}}{p_l^{(t)}} (\Delta p_l)^2 = \sum_{l \in [m]} \left( \sum_{k \in [m]} d_{ik}^{(t)} p_k^{(t)} \right) \frac{d_{il}^{(t)}}{p_l^{(t)}} (\Delta p_l)^2\]

\[= \sum_{l \in [m]} \sum_{k \in [m]} d_{il}^{(t)} d_{ik}^{(t)} \frac{p_k^{(t)}}{p_l^{(t)}} (\Delta p_l)^2\]

\[= \sum_{l \in [m]} (d_{il}^{(t)})^2 (\Delta p_l)^2 + \sum_{l \in [m]} \sum_{k \neq l} d_{il}^{(t)} d_{ik}^{(t)} \frac{p_k^{(t)}}{p_l^{(t)}} (\Delta p_l)^2\]

\[= \sum_{l \in [m]} (d_{il}^{(t)})^2 (\Delta p_l)^2 + \sum_{k \in [m]} \sum_{l \leq k} d_{ik}^{(t)} d_{il}^{(t)} \left( \frac{p_k^{(t)}}{p_l^{(t)}} (\Delta p_l)^2 + \frac{p_l^{(t)}}{p_k^{(t)}} (\Delta p_k)^2 \right)\]

Now, we apply the AM-GM inequality:

\[b_i \sum_{l \in [m]} \frac{d_{il}^{(t)}}{p_l^{(t)}} (\Delta p_l)^2 \geq \sum_{l \in [m]} (d_{il}^{(t)})^2 (\Delta p_l)^2 + \sum_{k < l} d_{ik}^{(t)} d_{il}^{(t)} (2|\Delta p_l||\Delta p_k|)\]

\[= \sum_{j \in [m]} \sum_{k \in [m]} d_{ij}^{(t)} d_{ik}^{(t)} |\Delta p_j| |\Delta p_k|\]

**An important result in microeconomics is the law of demand** which states that when the price of a good increases, the Hicksian demand for that good decreases in a very general setting of utility functions [53, 55]. We state a weaker version of the law of demand which is re-formulated to fit the tâtonnement framework.

**Lemma A.6 (Law of Demand).** [53, 55] Suppose that \( \forall j \in [m], t \in \mathbb{N}, p_j^{(t)}, p_j^{(t+1)} \geq 0 \) and \( u_1 \) is continuous and concave. Then, \( \sum_{j \in [m]} \Delta p_j \left( h_j^{(t+1)} - h_j^{(t)} \right) \leq 0 \).

A simple corollary of the law of demand which is used throughout the rest of this paper is that, during tâtonnement, the change in expenditure of the next time period is always less than or equal to the change in expenditure of the previous time period’s.

**Corollary A.7.** Suppose that \( \forall t \in \mathbb{N}, j \in [m], p_j^{(t)}, p_j^{(t+1)} \geq 0 \) and \( u_1 \) is continuous and concave then \( \forall t \in \mathbb{N}, \sum_{j \in [m]} \Delta p_j h_j^{(t+1)} \leq \sum_{j \in [m]} \Delta p_j h_j^{(t)} \).

The following lemma simply restates an essential fact about expenditure functions and Hicksian demand, namely that the Hicksian demand is the minimizer of the expenditure function.
**Lemma A.8.** Suppose that \( \forall j \in [m], p_j^{(t)}, t \in \mathbb{N}, h_{ij}^{(t)}, h_{ij}^{(t+1)} \geq 0 \) and \( u_i \) is continuous and concave then \( \sum_{j \in [m]} h_{ij}^{(t)} p_j^{(t)} \leq \sum_{j \in [m]} h_{ij}^{(t+1)} p_j^{(t)} \).

**Lemma A.8.** For the sake of contradiction, assume that \( \sum_{j \in [m]} h_{ij}^{(t)} p_j^{(t)} > \sum_{j \in [m]} h_{ij}^{(t+1)} p_j^{(t)} \). By the definition of the Hicksian demand, we know that the bundle \( h_i^{(t)} \) provides the buyer with one unit of utility. Recall that the expenditure at any price \( p \) is equal to the sum of the product of the Hicksian demands and prices, that is \( e_i(p, 1) = \sum_{j \in [m]} h_{ij}(p, 1)p_j \). Hence, we have \( e_i(p_j^{(t)}, 1) = \sum_{j \in [m]} h_{ij}^{(t)} p_j^{(t)} > \sum_{j \in [m]} h_{ij}^{(t+1)} p_j^{(t)} = e_i(p^{(t)}, 1) \), a contradiction.

We now introduce the following lemma which makes use of results on the behavior of Hicksian demand and expenditure functions in homothetic Fisher markets introduced by Goktas et al. [40]. In conjunction with Corollary A.7 and Lemma A.8 are key in proving that Lemma 4.1 holds allowing us to establish convergence of tâtonnement in a general setting of utility functions. Additionally, the lemma relates the Marshallian demand of homogeneous utility functions to their Hicksian demand. Before we present the lemma, we recall the following identities [55]:

\[
\forall b_i \in \mathbb{R}_+ \quad e_i(p, v_i(p, b_i)) = b_i \\
\forall v_i \in \mathbb{R}_+ \quad v_i(p, e_i(p, v_i)) = v_i \\
\forall b_i \in \mathbb{R}_+ \quad h_i(p, v_i(p, b_i)) = d_i(p, b_i) \\
\forall v_i \in \mathbb{R}_+ \quad d_i(p, e_i(p, v_i)) = h_i(p, v_i)
\]

**Lemma A.9.** Suppose that \( u_i \) is continuous and homogeneous, then the following holds:

\[\forall j \in [m] \quad d_{ij}(p, b_i) = \frac{b_i h_{ij}(p, 1)}{\sum_{j \in [m]} h_{ij}(p, 1)p_j} \]

**Lemma A.9.** We note that when utility function \( u_i \) is strictly concave, the Marshallian and Hicksian demand are unique making the following equalities well-defined.

\[
\frac{b_i h_{ij}(p, 1)}{\sum_{j \in [m]} h_{ij}(p, 1)p_j} = \frac{b_i h_{ij}(p, 1)}{e_i(p, 1)} \quad \text{(Definition of expenditure function)} \\
= b_i v_i(p, h_{ij}(p, 1)) \quad \text{(Corollary 1 of Goktas et al. [40])} \\
= v_i(p, b_i) h_{ij}(p, 1) \\
= h_{ij}(p, v_i(p, b_i)) \\
= d_{ij}(p, b_i) \quad \text{(Marshallian Demand Identity Equation (23))}
\]

We note that when utility function \( u_i \) is strictly concave, the Marshallian and Hicksian demand are unique making the following equalities well-defined.

\[\forall j \in [m], \frac{|\Delta p_j|}{p_j} \leq \frac{1}{4}, \text{ then for any } t \in \mathbb{N}_+ \text{ and } i \in [n]:
\]

\[
\frac{\left| h_i(p^{(t+1)}, 1), p^{(t+1)} - h_i(p^{(t)}, 1), p^{(t)} \right|}{\langle h_i^{(t)}, p^{(t)} \rangle} \leq \frac{1}{4}
\]

(25)
Proof. Case 1: \( \langle h_i(p^{(t+1)}, 1), p^{(t+1)} \rangle \geq \langle h_i(p^{(t)}, 1), p^{(t)} \rangle \)

\[
\frac{\langle h_i(p^{(t+1)}, 1), p^{(t+1)} \rangle - \langle h_i(p^{(t)}, 1), p^{(t)} \rangle}{\langle h_i(p^{(t)}, 1), p^{(t)} \rangle} 
\leq \frac{\langle h_i(p^{(t)}, 1), p^{(t+1)} \rangle - \langle h_i(p^{(t)}, 1), p^{(t)} \rangle}{\langle h_i(p^{(t)}, 1), p^{(t)} \rangle}
\]

(Corollary A.7)

\[
= \frac{\langle h_i(p^{(t)}, 1), p^{(t+1)} \rangle}{\langle h_i(p^{(t)}, 1), p^{(t)} \rangle} - 1
\]

\[
\leq \frac{5}{4} \frac{\langle h_i(p^{(t)}, 1), p^{(t+1)} \rangle}{\langle h_i(p^{(t)}, 1), p^{(t)} \rangle} - 1
\]

\[
= \frac{1}{4}
\]

where the penultimate line follows from the assumption that \( \forall j \in [m], \frac{|\Delta p_j|}{p_j} \leq \frac{1}{4} \).

Case 2: \( \langle h_i(p^{(t+1)}, 1), p^{(t+1)} \rangle \leq \langle h_i(p^{(t)}, 1), p^{(t)} \rangle \)

\[
\frac{\langle h_i(p^{(t)}, 1), p^{(t)} \rangle - \langle h_i(p^{(t+1)}, 1), p^{(t+1)} \rangle}{\langle h_i(p^{(t)}, 1), p^{(t)} \rangle} = 1 - \frac{\langle h_i(p^{(t+1)}, 1), p^{(t+1)} \rangle}{\langle h_i(p^{(t)}, 1), p^{(t)} \rangle}
\]

\[
\leq 1 - \frac{3}{4} \frac{\langle h_i(p^{(t)}, 1), p^{(t)} \rangle}{\langle h_i(p^{(t)}, 1), p^{(t)} \rangle}
\]

(Corollary A.7)

\[
= \frac{1}{4}
\]

where the second line follows from the assumption that \( \forall j \in [m], \frac{|\Delta p_j|}{p_j} \leq \frac{1}{4} \). □

Lemma A.11. Suppose that for all \( j \in [m], \frac{|\Delta p_j|}{p_j} \leq \frac{1}{4}, \) then for some \( c \in (0, 1) \) and \( t \in \mathbb{N}_+ \), we have:

\[
\frac{1}{b_1} \left( \langle d_i^{(t)}, \Delta p \rangle + \frac{b_1}{2} \left( \nabla_{\Delta p_j} e_i(p^{(t)} + c\Delta p, 1)\Delta p \right) e_i(p^{(t)}, 1) \right)^2
\]

\[
\leq \left( 1 + \frac{5c}{9} \right) \sum_{l \in [m]} \frac{d_i^{(t)}(\Delta p)_l}{p_l} (\Delta p_l)^2 + \left( \frac{25c^2}{432} \right) \sum_{l \in [m]} \frac{d_i^{(t)}(p^{(t)} + c\Delta p, b_l)}{p_l^{(t)}} (\Delta p_l)^2
\]

Proof.

\[
\frac{1}{b_1} \left( \langle d_i^{(t)}, \Delta p \rangle + \frac{b_1}{2} \left( \nabla_{\Delta p_j} e_i(p^{(t)} + c\Delta p, 1)\Delta p \right) e_i(p^{(t)}, 1) \right)^2
\]

(37)
where we denote $|\Delta p| = |\Delta p_1, \ldots, \Delta p_m|$.

\[
\leq \frac{1}{b_l} \left[ \left( d_i^{(t)}, |\Delta p| \right)^2 + \frac{5\epsilon}{3} \left( d_i^{(t)}, |\Delta p| \right) \left( \sum_j \frac{(\Delta p_j)^2}{p_j} d_{ij}(p^{(t)} + c\Delta p, b_i) \right) + \frac{25\epsilon^2}{36} \left( \sum_j \frac{(\Delta p_j)^2}{p_j} d_{ij}(p^{(t)} + c\Delta p, b_i) \right)^2 \right] 
\]

Since $\forall j \in [m], \frac{|\Delta p_j|}{p_j^{(t)}} \leq \frac{1}{4}$, we have:

\[
\leq \frac{1}{b_l} \left[ \left( d_i^{(t)}, |\Delta p| \right)^2 + \frac{5\epsilon}{12} \left( d_i^{(t)}, |\Delta p| \right) \left( \sum_j |\Delta p_j| d_{ij}(p^{(t)} + c\Delta p, 1) \right) + \frac{25\epsilon^2}{576} \left( \sum_j |\Delta p_j| d_{ij}(p^{(t)} + c\Delta p, b_i) \right)^2 \right] 
\]

\[
= \frac{1}{b_l} \left[ \sum_{i \in [m]} \sum_{k \in [m]} d_{ik}^{(t)} |\Delta p_j||\Delta p_k| + \frac{5\epsilon}{12} \sum_{i \in [m]} \sum_{k \in [m]} d_{ik}^{(t)} d_{ij}(p^{(t)} + c\Delta p, b_i) |\Delta p_k||\Delta p_j| + \frac{25\epsilon^2}{576} \sum_{i \in [m]} \sum_{k \in [m]} d_{ik}^{(t)} d_{ij}(p^{(t)} + c\Delta p, b_i) |\Delta p_k||\Delta p_j| \right] 
\]

\[
\leq \sum_{i \in [m]} \frac{d_i^{(t)}}{p_i^{(t)}} (\Delta p_i)^2 + \frac{1}{b_l} \frac{5\epsilon}{12} \sum_{i \in [m]} \sum_{k \in [m]} d_{ik}^{(t)} d_{ij}(p^{(t)} + c\Delta p, b_i) |\Delta p_k||\Delta p_j| + \frac{25\epsilon^2}{576} \sum_{i \in [m]} \sum_{k \in [m]} d_{ik}^{(t)} d_{ij}(p^{(t)} + c\Delta p, b_i) |\Delta p_k||\Delta p_j| 
\]
where the last line was obtained by (Lemma A.5). Continuing, by Lemma A.4, we have:
\[
\leq \sum_{i\in[m]} \frac{d_i(t)}{p_i(t)} (\Delta p_i)^2 + \frac{5\epsilon}{12} \sum_{i\in[m]} \frac{d_i(t)}{p_i(t)} (\Delta p_i)^2 + \frac{25\epsilon^2}{576} \sum_{i\in[m]} \frac{d_i(p(t) + c\Delta p_i b_i)}{p_i(t)} (\Delta p_i)^2
\]
\[
= \left(1 + \frac{5\epsilon}{9}\right) \sum_{i\in[m]} \frac{d_i(t)}{p_i(t)} (\Delta p_i)^2 + \frac{25\epsilon^2}{432} \sum_{i\in[m]} \frac{d_i(p(t) + c\Delta p_i b_i)}{p_i(t)} (\Delta p_i)^2
\]

(46)  

(47)

**Lemma A.12.** Suppose that \( \frac{|\Delta p_j|}{p_j} \leq \frac{1}{4} \), then
\[
b_i \log \left(1 - \frac{e_i(p(t+1), 1) - e_i(p(t), 1)}{e_i(p(t), 1)} \left(1 + \frac{e_i(p(t+1), 1) - e_i(p(t), 1)}{e_i(p(t), 1)} \right)^{-1}\right)
\]
\[
\leq \left(1 + \frac{20\epsilon}{27}\right) \sum_{i\in[m]} \frac{d_i(t)}{p_i(t)} (\Delta p_i)^2 + \frac{5\epsilon}{6} \sum_{i\in[m]} \frac{(\Delta p_i)^2}{p_i(t)} - d_i(p(t), 1) - \left(d_i(p(t), 1), \Delta p\right)
\]

**Lemma A.12.** First, we note that \( h_i^1 \cdot p(t) > 0 \) because prices during our tâtonnement rule reach \( 0 \) only asymptotically and Hicksian demand for one unit of utility at prices \( p(t) > 0 \) is strictly positive; and likewise, prices reach \( \infty \) only asymptotically, which implies that Hicksian demand is always strictly positive. This fact will come handy, as we divide some expressions by \( h_i^1 \cdot p(t) \).

Fix \( t \in \mathbb{N}_+ \) and \( i \in [n] \). Since by our assumptions \( \frac{|\Delta p_j|}{p_j} \leq \frac{1}{4} \), by Lemma A.10, we have \( 0 \leq \frac{e_i(p(t+1), 1) - e_i(p(t), 1)}{e_i(p(t), 1)} \leq \frac{1}{4} \). We can then use the bound \( 1 - x(1 + x)^{-1} \leq 1 + \frac{4}{3}x^2 - x \), for \( 0 \leq |x| \leq \frac{1}{4} \), with \( x = \frac{e_i(p(t+1), 1) - e_i(p(t), 1)}{e_i(p(t), 1)} \), to get:
\[
b_i \log \left(1 - \frac{e_i(p(t+1), 1) - e_i(p(t), 1)}{e_i(p(t), 1)} \left(1 + \frac{e_i(p(t+1), 1) - e_i(p(t), 1)}{e_i(p(t), 1)} \right)^{-1}\right)
\]
\[
\leq b_i \log \left(1 + \frac{4}{3} \left(\frac{e_i(p(t+1), 1) - e_i(p(t), 1)}{e_i(p(t), 1)}\right)^2 - \frac{e_i(p(t+1), 1) - e_i(p(t), 1)}{e_i(p(t), 1)}\right)
\]

Let \( a = \frac{4}{3} \left(\frac{e_i(p(t+1), 1) - e_i(p(t), 1)}{e_i(p(t), 1)}\right)^2 - \frac{e_i(p(t+1), 1) - e_i(p(t), 1)}{e_i(p(t), 1)} \). By Lemma A.10, we know that \( 0 + (-1/4) \leq a \leq \frac{1}{12} + 1/4 \Leftrightarrow a \leq \frac{1}{4} \). We now use the bound \( x \geq \log (1 + x) \) for \( x > -1 \), with \( x = a \) to get:
\[
b_i \log \left(1 + \frac{4}{3} \left(\frac{e_i(p(t+1), 1) - e_i(p(t), 1)}{e_i(p(t), 1)}\right)^2 - \frac{e_i(p(t+1), 1) - e_i(p(t), 1)}{e_i(p(t), 1)}\right)
\]
\[
\leq b_i \log \left(1 + \frac{4}{3} \left(\frac{e_i(p(t+1), 1) - e_i(p(t), 1)}{e_i(p(t), 1)}\right)^2 - \frac{e_i(p(t+1), 1) - e_i(p(t), 1)}{e_i(p(t), 1)}\right)
\]

Using a first order Taylor expansion of \( e_i(p(t) + \Delta p, 1) \) around \( p(t) \), by Taylor’s theorem [42], we have: \( e_i(p(t) + \Delta p, 1) = e_i(p(t), 1) + \langle \nabla p e_i(p(t), 1), \Delta p \rangle + \frac{1}{2} \langle \nabla^2 p e_i(p(t), 1) + c\Delta p, 1 \rangle \Delta p, 1 \Delta p, 1 \) for some \( c \in (0, 1) \). Re-organizing terms around, we get \( e_i(p(t+1), 1) - e_i(p(t), 1) = \langle \nabla p e_i(p(t), 1), \Delta p \rangle + \frac{1}{2} \langle \nabla^2 p e_i(p(t), 1) + c\Delta p, 1 \rangle \Delta p, 1 \Delta p, 1 \).
\[
\frac{1}{2}\left(\nabla^2 e_i(p^{(t)}) + c\Delta p, 1\Delta p, \Delta p\right), \text{ which gives us:}
\]
\[
= b_i \left(\frac{4}{3} \left( \frac{\langle \nabla^2 e_i(p^{(t)}, 1), \Delta p \rangle + \frac{1}{2} \left( \nabla^2 e_i(p^{(t)} + c\Delta p, 1)\Delta p \right)}{e_i(p^{(t)}, 1)} \right)^2 \right) - \frac{\langle h_i(p^{(t)}, 1), \Delta p \rangle + \frac{1}{2} \left( \nabla^2 e_i(p^{(t)} + c\Delta p, 1)\Delta p \right)}{e_i(p^{(t)}, 1)}
\]
\[
= \frac{4}{3} b_i \left( \frac{b_i \langle h_i(p^{(t)}, 1), \Delta p \rangle}{e_i(p^{(t)}, 1)} + \frac{b_{i/2} \left( \nabla^2 e_i(p^{(t)} + c\Delta p, 1)\Delta p \right)}{e_i(p^{(t)}, 1)} \right)^2 \right) - \frac{b_i \langle h_i(p^{(t)}, 1), \Delta p \rangle}{e_i(p^{(t)}, 1)} - \frac{b_{i/2} \left( \nabla^2 e_i(p^{(t)} + c\Delta p, 1)\Delta p \right)}{e_i(p^{(t)}, 1)}
\]
\[
= \frac{4}{3} b_i \left( \frac{\langle d_i(p^{(t)}, 1), \Delta p \rangle}{e_i(p^{(t)}, 1)} + \frac{b_{i/2} \left( \nabla^2 e_i(p^{(t)} + c\Delta p, 1)\Delta p \right)}{e_i(p^{(t)}, 1)} \right)^2 \right) - \frac{\langle d_i(p^{(t)}, 1), \Delta p \rangle}{e_i(p^{(t)}, 1)} - \frac{b_{i/2} \left( \nabla^2 e_i(p^{(t)} + c\Delta p, 1)\Delta p \right)}{e_i(p^{(t)}, 1)}
\]

Continuing, by Shepherd’s lemma [67], we have:

\[
\frac{4}{3} b_i \left( \frac{\langle h_i(p^{(t)}, 1), \Delta p \rangle}{e_i(p^{(t)}, 1)} + \frac{b_{i/2} \left( \nabla^2 e_i(p^{(t)} + c\Delta p, 1)\Delta p \right)}{e_i(p^{(t)}, 1)} \right)^2 \right) - \frac{b_i \langle h_i(p^{(t)}, 1), \Delta p \rangle}{e_i(p^{(t)}, 1)} - \frac{b_{i/2} \left( \nabla^2 e_i(p^{(t)} + c\Delta p, 1)\Delta p \right)}{e_i(p^{(t)}, 1)}
\]

where the last line was obtained from Lemma A.9.

Using Lemma A.11, we have:

\[
\frac{4}{3} \left( \frac{5}{9} \sum_{l \in [m]} \frac{d_{il}^{(t)}}{p_l^{(t)}} \frac{\Delta p_l^2}{\Delta p_l} \frac{\Delta p_l}{\Delta p_l} - \frac{1}{2} \left( \Delta p_l \right)^2 \right) - \frac{b_i \langle h_i(p^{(t)}, 1), \Delta p \rangle}{e_i(p^{(t)}, 1)} - \frac{b_{i/2} \left( \nabla^2 e_i(p^{(t)} + c\Delta p, 1)\Delta p \right)}{e_i(p^{(t)}, 1)}
\]

\[
= \frac{4}{3} \left( \frac{20}{27} \sum_{l \in [m]} \frac{d_{il}^{(t)}}{p_l^{(t)}} \frac{\Delta p_l^2}{\Delta p_l} - \frac{1}{2} \left( \Delta p_l \right)^2 \right) - \frac{b_i \langle h_i(p^{(t)}, 1), \Delta p \rangle}{e_i(p^{(t)}, 1)} - \frac{b_{i/2} \left( \nabla^2 e_i(p^{(t)} + c\Delta p, 1)\Delta p \right)}{e_i(p^{(t)}, 1)}
\]

\[
= \frac{4}{3} \left( \frac{20}{27} \sum_{l \in [m]} \frac{d_{il}^{(t)}}{p_l^{(t)}} \frac{\Delta p_l^2}{\Delta p_l} - \frac{1}{2} \left( \Delta p_l \right)^2 \right) - \frac{b_i \langle h_i(p^{(t)}, 1), \Delta p \rangle}{e_i(p^{(t)}, 1)} - \frac{b_{i/2} \left( \nabla^2 e_i(p^{(t)} + c\Delta p, 1)\Delta p \right)}{e_i(p^{(t)}, 1)}
\]
Finally, we note that $\nabla^2_{\rho} e_i$ is negative semi-definite, meaning that we have $\left< \nabla^2_{\rho} e_i(p^{(t)}) + c\Delta p, \Delta p \right> \leq 0$, allowing us to re-express Equation (53) as follows:

$$
= \left( \frac{4}{3} + \frac{20\epsilon}{27} \right) \sum_{l \in [m]} \frac{d_{il}^{(t)}}{p_l^{(t)}} (\Delta p_l)^2 + \frac{25\epsilon^2}{324} \sum_{l \in [m]} \frac{(\Delta p_l)^2}{p_l^{(t)}} d_{il}(p^{(t)}) + c\Delta p, b_l - \left< d_i(p^{(t)}, 1), \Delta p \right> + \frac{5\epsilon}{6} \sum_{l \in [m]} \frac{(\Delta p_l)^2}{p_l^{(t)}} d_{il}(p^{(t)}) + c\Delta p, b_l)
$$

$$
\leq \left( \frac{4}{3} + \frac{20\epsilon}{27} \right) \sum_{l \in [m]} \frac{d_{il}^{(t)}}{p_l^{(t)}} (\Delta p_l)^2 + \frac{25\epsilon^2}{324} \sum_{l \in [m]} \frac{(\Delta p_l)^2}{p_l^{(t)}} d_{il}(p^{(t)}) + c\Delta p, b_l - \left< d_i(p^{(t)}, 1), \Delta p \right>
$$

where the penultimate line was obtained from Lemma 3.2.

□

**Lemma 4.3 (Step size for homothetic complements Markets).** If tâtonnement is run on a complements Fisher market $(u, b)$, then, for all $t \in \mathbb{N}_+$, prices satisfy $\left| \frac{\Delta p_j}{p_j^{(t)}} \right| \leq \frac{1}{3}$ if and only if

$$
y \geq \max \left\{ d_{ij}^{(0)}, \max_{p \in \Delta_m} \max_{k \in [m]} \frac{h_{ik}(p, 1)}{h_{ij}(p, 1)} \right\}.
$$

**Proof of Lemma 4.3.** ($\Rightarrow$) Suppose that $d_{ij}^{(t)} \geq \max_{p \in \Delta_m} \max_{k \in [m]} \frac{h_{ik}(p, 1)}{h_{ij}(p, 1)}$. Then, for all $k \in [m]$:

$$
d_{ik}^{(t)} = \frac{h_{ik}^{(t)}}{h_{ij}^{(t)}} d_{ij}^{(t)}
$$

$$
\geq \frac{h_{ik}^{(t)}}{h_{ij}^{(t)}} \max_{p \in \Delta_m} \max_{l \in [m]} \frac{h_{ij}(p, 1)}{h_{il}(p, 1)}
$$

$$
\geq \frac{h_{ik}^{(t)}}{h_{ij}^{(t)}} = 1
$$

This means that the price of good $j$ will increase in the next time period, i.e., $\forall j \in [m], p_{j}^{(t+1)} \geq p_{j}^{(t)}$ which implies that $e_i(p^{(t+1)}, 1) \geq e_i(p^{(t)}, 1)$. Which gives us:

$$
\frac{b_l}{e_i(p^{(t+1)}, 1)} \leq \frac{b_l}{e_i(p^{(t)}, 1)}
$$
\[
\begin{align*}
\frac{b_i h_{ij}^{(t+1)}}{e_i(p^{(t+1)}, 1)} & \leq \frac{b_i h_{ij}^{(t+1)}}{e_i(p^{(t)}, 1)} \\
\frac{d_{ij}^{(t+1)}}{b_i h_{ij}^{(t+1)}} & \leq \frac{d_{ij}^{(t+1)}}{e_i(p^{(t)}, 1)} \\
\frac{d_{ij}^{(t+1)}}{b_i h_{ij}^{(t+1)}} & = \frac{d_{ij}^{(t)}}{e_i(p^{(t)}, 1)} = d_{ij}^{(t)}
\end{align*}
\] (61)

(62)

(63)

where the last line follows from the net substitutes assumption, since all prices increase at time 
\( t + 1 \).

Now, instead suppose that \( d_{ij}^{(t)} \leq \max_{p \in \Delta_m} \max_{k \in [m]} \max_{h_{ik} (p, 1) > 0} \left\{ \frac{h_{ij} (p, 1)}{h_{ik} (p, 1)} \right\} \), then the demand for good \( j \) increases the most when the prices of all goods go down, which by our assumption that 
\( \forall j \in [m], \frac{\Delta p_j}{p_j} \) gives us:

\[
\begin{align*}
d_{ij} (p^{(t+1)}, b_i) & \leq d_{ij} (3/4 p^{(t)}, b_i) \\
& \leq 4/3 d_{ij} (p^{(t)}, b_i) \\
& \leq \max_{p \in \Delta_m} \max_{k \in [m]} \max_{h_{ik} (p, 1) > 0} \left\{ \frac{h_{ij} (p, 1)}{h_{ik} (p, 1)} \right\}
\end{align*}
\] (64)

(65)

(66)

where the last line is obtained by the homogeneity of degree -1 of Marshallian demand in \( p \) for homothetic preferences [40] and by our assumption. Then, by induction on \( t \), we obtain:

\[
\begin{align*}
d_{ij}^{(t)} & \leq \max \left\{ d_{ij}^{(t)}, \max_{p \in \Delta_m} \max_{k \in [m]} \max_{h_{ik} (p, 1) > 0} \left\{ \frac{h_{ij} (p, 1)}{h_{ik} (p, 1)} \right\} \right\}
\end{align*}
\] (67)

\( \iff \)

The opposite direction follows by induction on the time steps of tâtonnement. On iteration 0, the condition holds trivially, then by the induction hypothesis can be proven by the same argument as above.

\( \square \)