CS256
Applied Theory of Computation

VLSI Model IV

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Overview

- Completion of Planar Separator Theorem

Area-Time Computational Inequalities

$$C_p(f) = O(\min(AT^2, A^2T))$$

- To derive lower bounds on $C_p(f)$ we introduce the **planar separator theorem**. In its simplest form, it states that the vertices in every planar n-vertex graph can be divided into two sets with no edges between them by the removal of $O(\sqrt{n})$ vertices such that each set has between $n/3$ and $2n/3$ vertices.
- We use it to show that some functions have a quadratic planar circuit size in their number of inputs. The technique used is to show that a lot of information must pass from inputs to outputs.

Planar Separator Theorem

- Let G = (V,E) and let *c* : V → *R* assign non-negative costs to vertices. The cost of a subset S of V is the sum of the cost of the elements of S.

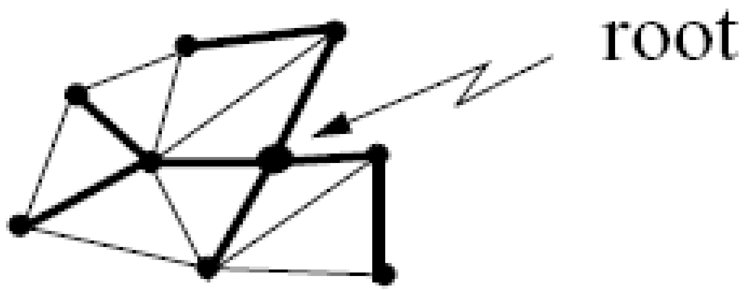
  Lemma If G has a rooted spanning tree of radius *r*, V can be partitioned into disjoint sets A, B, C such that *c*(A), *c*(B) ≤ 2*c*(V)/3, no edge joins vertices in A and B, and C contains at most $2r+1$ vertices.

Planar Separator Theorem

Theorem I Let G = (V,E) be an *N*-vertex planar graph having non-negative vertex costs summing to *c*(V). Then, V can be partitioned into three sets, A, B, and C, such that no edge joins vertices in A with those in B, neither A nor B has cost exceeding 2*c*(V)/3, and C contains no more than $4\sqrt{N}$ vertices.

Planar Separator Theorem

Proof We assume G is connected. If not, embed it in the plane and add edges as appropriate to make it connected. Assume that it has been triangulated. Pick any vertex (call it the root) and perform a breadth-first traversal of G. This traversal defines a **BFS spanning tree T** of G.



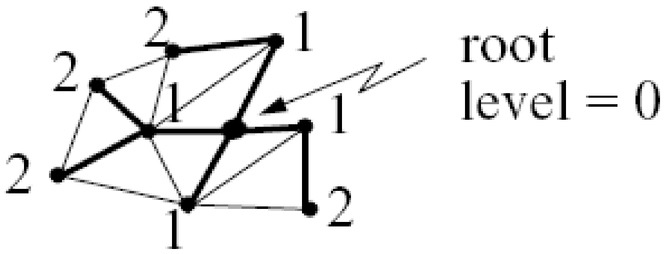
Planar Separator Theorem

A vertex v has level d in this tree if the length of the path from the root to v has d edges. There are no vertices at level q where q is the level one larger than that of all vertices ($q = 3$ in the example). Let R_d be the vertices at level d and let $r_d = |R_d|$.

Planar Separator Theorem

In Problem 12.9 it is stated that there is some level m such that the cost of vertices at levels below and above m each is at most $c(V)/2$.

Let l and h, $l \le m \le h$, be levels closest to m that contain at most $\sqrt{N}$ vertices. That is, r_l , $r_h \le \sqrt{N}$. There are such levels because level 0 contains a single vertex and there are none at level q.



Planar Separator Theorem

Vertices in G are partitioned into five sets: a) $H = \cup_{d < l} R_d$ (*high vertices* close to the root), b) R_l (vertices at level *l*), c) $M = \cup_{l < d < h} R_d$ (*middle vertices*), d) R_h, e) $L = \cup_{h < d} R_d$ (*low vertices*).

Because *L* and *H* are subsets of the vertices with levels less than and more than *m*, $c(L)$, $c(H) \leq c(V)/2$. By construction, $r_l, r_h \leq \sqrt{N}$.

Planar Separator Theorem

If $R_l = R_h = R_m$ (which implies that M is empty), let A = L, B = H, and C = $R_l = R_h$.

Then, C is a separator of size at most $\sqrt{N}$ and the theorem holds. If $l \neq h$, then $h - l - 1 \geq 0$.

Since each of the $h - l - 1$ levels between r_l and r_h has at least $\sqrt{N}+1$ vertices, it follows that $h - l - 1 \leq \sqrt{N} - 1$ because these levels have $\leq N-1$ vertices.

Planar Separator Theorem

Consider the subgraph of G consisting of the vertices in *M* and the edges between them. Add a new vertex v_0 to replace the vertices in $H \cup R_l$ and add an edge from v_0 to each of the vertices at level $l+1$. This operation retains planarity and the resulting graph remains triangulated because adjacent vertices on R_{l+1} have an edge between them. Also, it defines a spanning tree T* consisting of v_0, the new edges, and the projection of the original spanning tree to the vertices in *M*. T*has radius at most $\sqrt{N}$.

Planar Separator Theorem

Apply Lemma of last lecture to T* while giving v_0 zero cost. The lemma identifies three sets of vertices, A_0, B_0 and C_0, from which we delete v_0. Since $c(M) \leq c(V)$, it follows that there are no edges between vertices in A_0 and B_0, $c(A_0)$, $c(B_0) \leq 2c(V)/3$, and $|C_0| \leq 2\sqrt{N}$. Let $C = C_0 \cup R_l \cup R_h$. Thus, $|C| \leq 4\sqrt{N}$.

Planar Separator Theorem

Each of the four sets A_0, B_0, *L*, and *H* has cost at most 2*c*(V)/3. If any one of them has cost more than *c*(V)/3, let it be A; let B be the union of remaining sets. It follows that c(V)/3 ≤ c(A), c(B) ≤ 2c(V)/3.

If none of them has cost more than *c*(V)/3, order the sets by size and let A be the union of the fewest of these sets whose cost is ≥ *c*(V)/3 vertices. This procedure ensures that c(V)/3 ≤ c(A) ≤ 2c(V)/3 which implies that B satisfies the same condition and theorem is established. QED

Planar Separator Theorem

Theorem II Let G = (V,E) be an N-vertex planar graph with non-negative vertex costs summing to *c*(V). Then V can be partitioned into three sets, A, B, and C, such that no edge joins vertices in A with those in B, neither A nor B has cost exceeding 7c(V)/9, |A|, |B| ≤ 5N/6, and C contains no more than $K_1\sqrt{N}$ vertices, where $K_1 = 4(\sqrt{(2/3)} + 1)$.

Planar Separator Theorem

Theorem III Let $G = (V,E)$ be an N-vertex planar graph and let c be a non-negative cost function on V with total cost of $c(V)$. Let $P \geq 2$. There are constants $2P/3 \leq q \leq 3P$ and $K_2 = 4(\sqrt{(2/3)} + 1)/(1 - \sqrt{(5/6)})$ such that V can be partitioned into q sets, $A_1, A_2, \ldots, A_q$ such that for $1 \leq i \leq q$

$$c(V)/(3P) \leq c(A_i) \leq 3c(V)/(2P)$$

and sets C_i, $|C_i| \leq K_2 \sqrt{N}$, and $B_i = V - A_i - C_i$ such that no edges join vertices in A_i with vertices in B_i.