CS256
Applied Theory of Computation

VLSI Model III

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Overview

- Planar Separator Theorem

Area-Time Computational Inequalities

$$C_p(f) = O(min(AT^2, A^2T))$$

- To derive lower bounds on $C_p(f)$ we introduce the **planar separator theorem**. In its simplest form, it states that the vertices in every planar *n*-vertex graph can be divided into two sets with no edges between them by the removal of $O(\sqrt{n})$ vertices such that each set has between *n/3* and *2n/3* vertices.
- We will use it to show that some functions have a quadratic planar circuit size in their number of inputs. The technique used is to show that a lot of information must pass from inputs to outputs.

Planar Separator Theorem

- Let G = (V,E) and let *c* : V → *R* assign non-negative costs to vertices. The cost of a subset S of V is the sum of the cost of the elements of S.

  Lemma If G has a rooted spanning tree of radius *r*, V can be partitioned into disjoint sets A, B, C such that *c*(A), *c*(B) ≤ 2*c*(V)/3, no edge joins vertices in A and B, and C contains at most $2r+1$ vertices.

Planar Separator Theorem

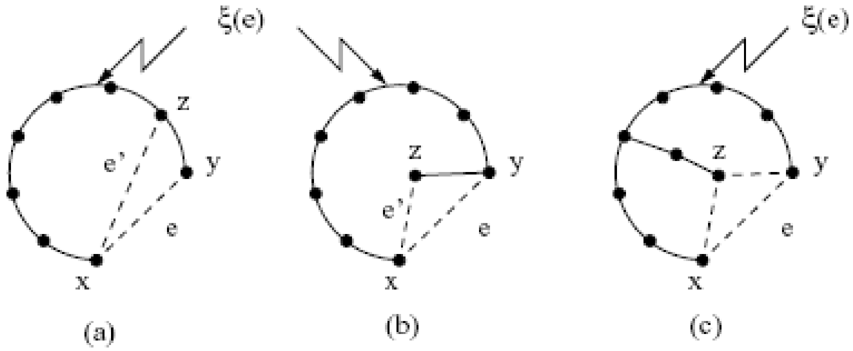
Proof Since the lemma is true if any vertex has cost > c(V)/3, assume the converse. Let G be embedded in the plane. Consider a face of the embedding, a region bounded by edges not containing other vertices or edges. (The graph has one external face of unbounded area.)

A **triangular planar graph** has only triangular faces. If G is not triangular, make it so by picking one vertex on each face and adding edges to all other vertices on the face.

Planar Separator Theorem

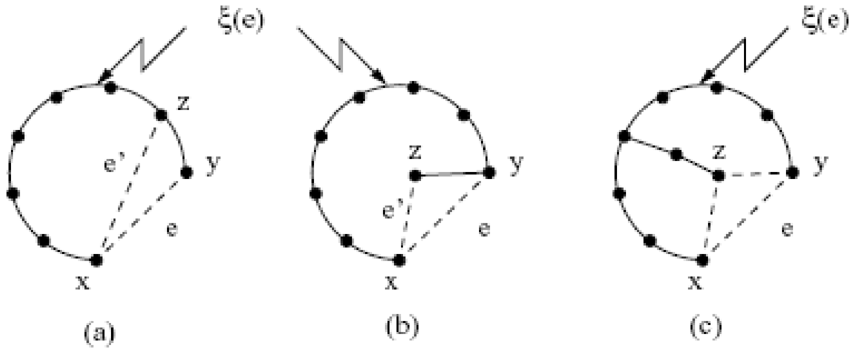
Proof (cont.) Let T be radius-*r* spanning tree of G. Each edge *e* in E not in T defines a unique cycle $\xi(e)$ of length at most $2r+1$ dividing V into vertices on $\xi(e)$ and those inside and outside $\xi(e)$. Let $c_1(e)$ and $c_2(e)$ be cost inside and outside $\xi(e)$ and let $\mu(e) = max(c_1(e), c_2(e))$. To show $\mu(e) \leq 2c(V)/3$, assume not. W.l.o.g. let $c_1(e) \geq c_2(e)$. (Side with larger cost is **inside**.)

Planar Separator Theorem



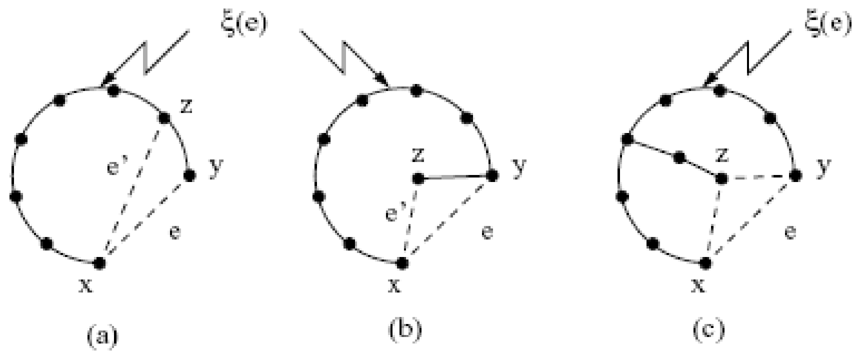
Proof (cont.) Let *e* = *(x,y)* in E not on T be such that μ(*e*) is minimal, and for all *e** such that μ(*e**) = μ(*e*), let inside of ξ(*e*) have fewer faces. In ties, choose *e* arbitrarily. We contradict assumption μ(*e*) > 2*c*(V)/3.

Planar Separator Theorem



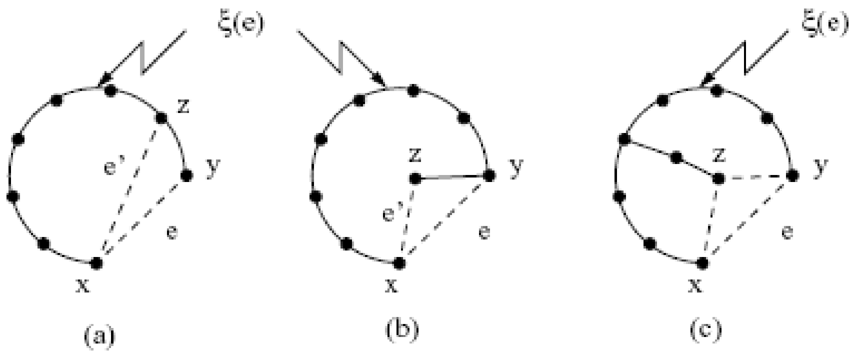
Proof (cont.) Consider triangle containing *e = (x,y)* on side with larger cost (the inside). Let *z* be third vertex in the triangle. It is on spanning tree (as are all vertices).

Planar Separator Theorem



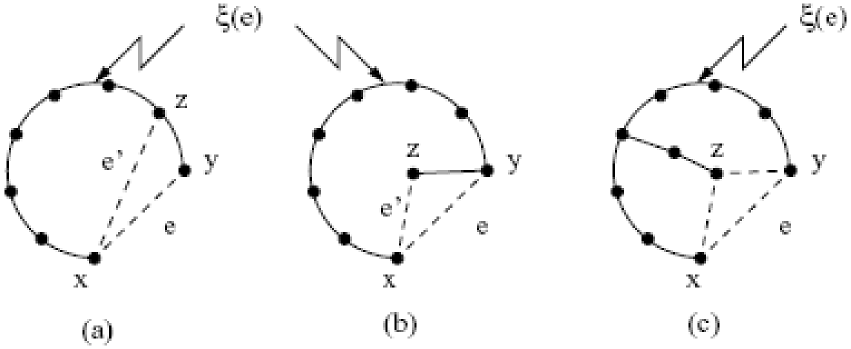
Two cases: i) either (x,z) or (y,z) is in T or ii) neither edge in T. In i) no loss in generality to assume (y,z) in T. Two subcases shown in Figure, a) z on $\xi(e)$, and b) z not on $\xi(e)$.

Planar Separator Theorem



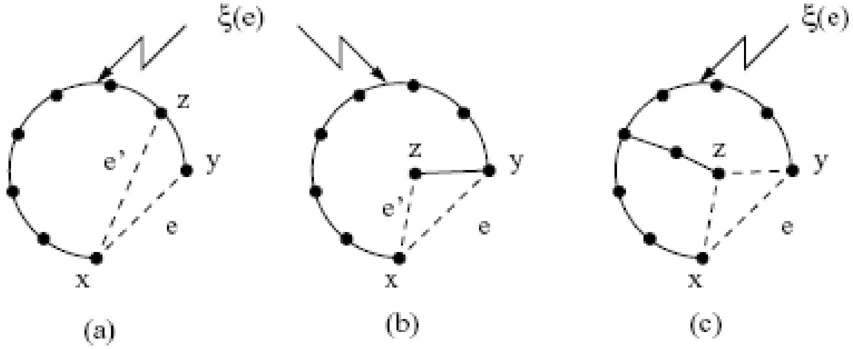
Proof (cont.) In a) $e' = (x,z)$ cannot be in T because T has no cycles unless x, y, and z form G; but then $\xi(e)$ has zero cost, contradicting assumption $\mu(e) > 2c(V)/3$. Also $\xi(e')$ has same set of vertices inside and out as does $\xi(e)$ but has fewer faces, contradicting definition of e.

Planar Separator Theorem



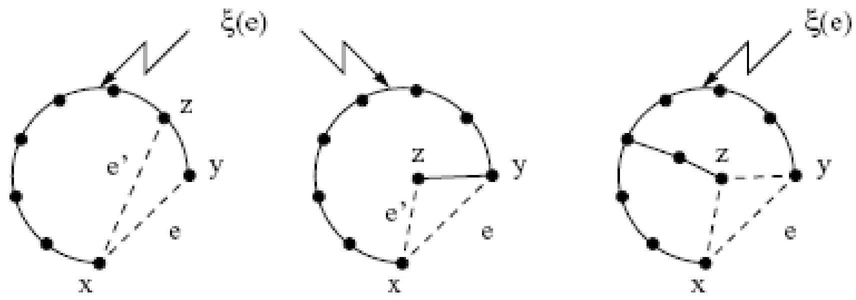
Proof (cont.) In case b) *e' = (x,z)* is non-tree edge (T has no cycles) Cost inside of ξ(*e'*) ≤ cost inside ξ(*e*) and one less face. If cost inside ξ(*e*') is greater than cost outside, it would have been chosen instead of *e*. On other hand, if cost inside ξ(*e*') is at most cost outside, since latter is equal to cost outside ξ(*e*), which is at most *c*(V)/3, cost inside ξ(*e*') is at most *c*(V)/3, which contradicts μ(*e**) > 2*c*(V)/3 for all *e**, so this case doesn't arise.

Planar Separator Theorem



Consider case ii) shown in c). Edges *(x,z)* and *(y,z)* each define a cycle inside one defined by *e*. W.l.o.g. assume cycle defined by *(x,z)*, ξ((x,z)), has greater cost inside than *(y,z)*, ξ((y,z)). Cost inside ξ(*e*) (which is > 2*c*(V)/3) is sum of the cost inside cycles ξ((x,z)) and ξ((y,z)) plus the cost *t* of vertices on the tree branch to *z*.

Planar Separator Theorem



(a) (b) (c)

It follows that the cost inside $\xi((x,z))$ plus $t/2$ is $> c(V)/3$. Thus, the cost outside $\xi((x,z))$ is $< 2c(V)/3$. If cost inside $\xi((x,z))$ is also $< 2c(V)/3$, contradicting $\mu(e^*) > 2c(V)/3$ for all e^*. If cost inside $\xi((x,z)) > 2c(V)/3$, $\xi((x,z))$ is a cycle with fewer faces than $\xi(e)$ with with cost $> 2c(V)/3$, a contradiction. Thus, all situations are impossible. QED