CS256
Applied Theory of Computation

Parallel Computation IV

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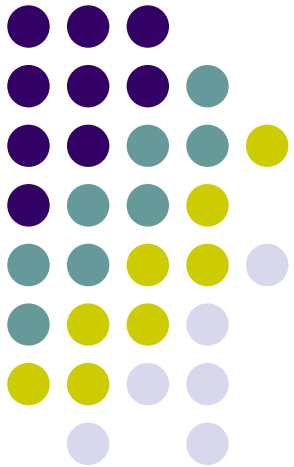
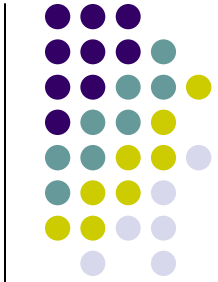


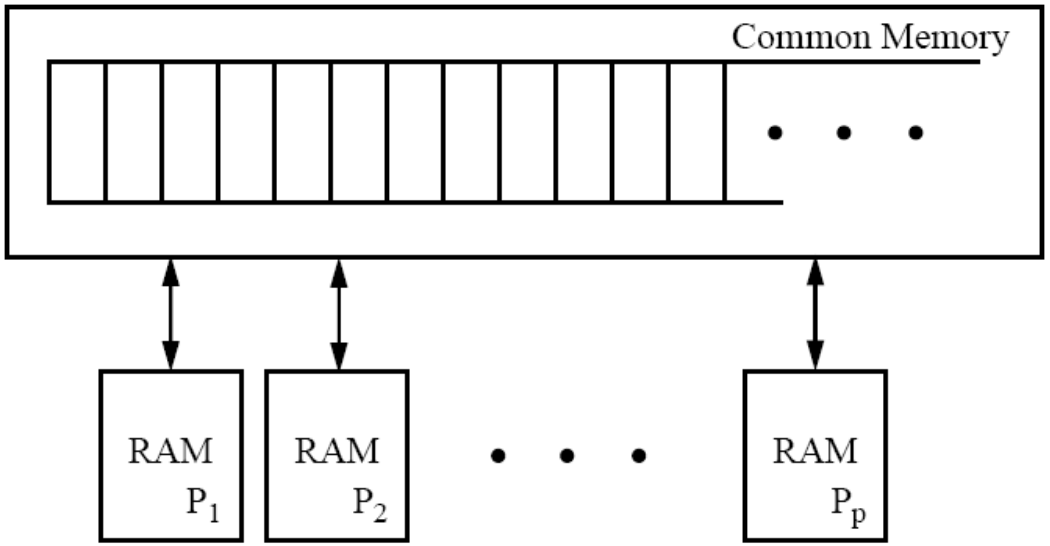
Overview



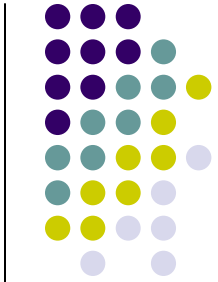
- PRAM
- Work-time framework for parallel algorithms
- Prefix computations
- Finding roots of trees in a forest
- Parallel merging
- Parallel partitioning
- Sorting
- Simulating CREW on EREW PRAM

PRAM Model

- The PRAM is an abstract programming model
  - Processors operate synchronously, reading from memory, executing locally, and writing to memory.



PRAM Model



- Four types: EREW, ERCW, CREW, CRCW
  - R – read, W – write, E – exclusive, C – common
- Can Boolean functions be computed quickly?
  - How can a function be represented?
  - Can we use concurrency to good advantage?
  - Is this use of concurrency realistic?
- Good source: Intro to Parallel Algs by Jaja

Matrix Multiplication on PRAM

Input: n×n matrices A and B

Output: n×n matrix C, local vars. C'(i,j,l), n

begin

Compute C(i,j,l) = A(i,l)B(l,j)

For h = 1 **to** $\log_2$ n **do**

if (l ≤ n) **then** C(i,j,l) := C(i,j,2l-1) + C'(i,j,2l)

If (l = 1) **then** C(i,j) := C(i,j,l)

End

- Running time on CREW is O(log n).
  - Why is CREW necessary?

Measuring Performance of Parallel Programs

- Let T(n) time and P(n) processors be used on a parallel machine on a problem with n inputs
- **Cost**: C(n) = P(n)T(n) is the time × processor product, or work, for problem on n inputs.
- An equivalent serial algorithm will run in time O(C(n)).
- If p ≤ P(n) processors available, we can implement the algorithm in time O(P(n)T(n)/p) or O(C(n)/p) time.

Advantages of PRAM

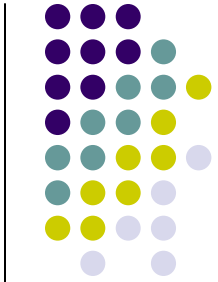
- A body of algorithms exist for this shared memory model.
- The model ignores algorithmic details of synchronization and communication.
- It makes explicit the association between operations and processors.
- PRAM algorithms are robust – network-based algorithms can be derived from them.
- PRAM is MIMD model.

Work-Time Framework for Parallel Algorithms

- Informal guideline to algorithm performance on PRAM.
- Work-time framework exhibits parallelism.
  - Use **for** $l \le i \le u$ **pardo** for parallel operations
  - Also allow serial straight-line and branching ops

- W(n) (**work**) is total no. of ops on n inputs
- T(n) is the running time of algorithm

Work-Time Framework for Parallel Algorithms



Sum

Input: n = 2^k inputs in array A[n]

Output: Sum S = A(1)+A(2)+ … + A(n)

begin

Copy A[n] to B[n]

for h = 1 **to** $\log_2$ n **do**

for $1 \le i \le n/2^h$ **pardo**

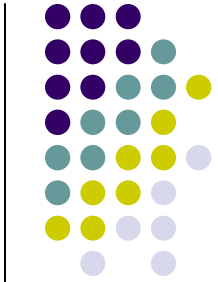
B(i) := B(2i-1) + B(2i)

S := B(1)

Work-Time Framework for Parallel Algorithms

- **Rescheduling:** Generally through rescheduling the following bounds can be achieved.
  - If p procs. available, time is $\leq W(n)/p + T(n)$.
  - Cost $C(n) = p(W(n)/p+T(n)) \leq O(W(n)+p\ T(n))$

When Is a Parallel Computation Optimal?



- If the smallest parallel work is about the same as the best serial time, the parallel machine cannot ran substantially faster.

Prefix Computation – Powerful Operation with Many Applications

Input: $(x_1,\ldots,x_n)$, $n = 2^k$ elements of S.

Output: $(s_1,\ldots,s_n)$, $s_i = x_1 * \ldots * x_i$, * an associative op

begin

If n = 1 **then** ($s_1 := x_1$; **exit**)

for $1 \le i \le n/2$ **pardo**

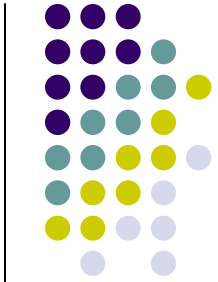
$y_i = x_{2i-1} * x_{2i}$

Compute prefix $(z_1,\ldots,z_{n/2})$ from $(y_1,\ldots,y_{n/2})$

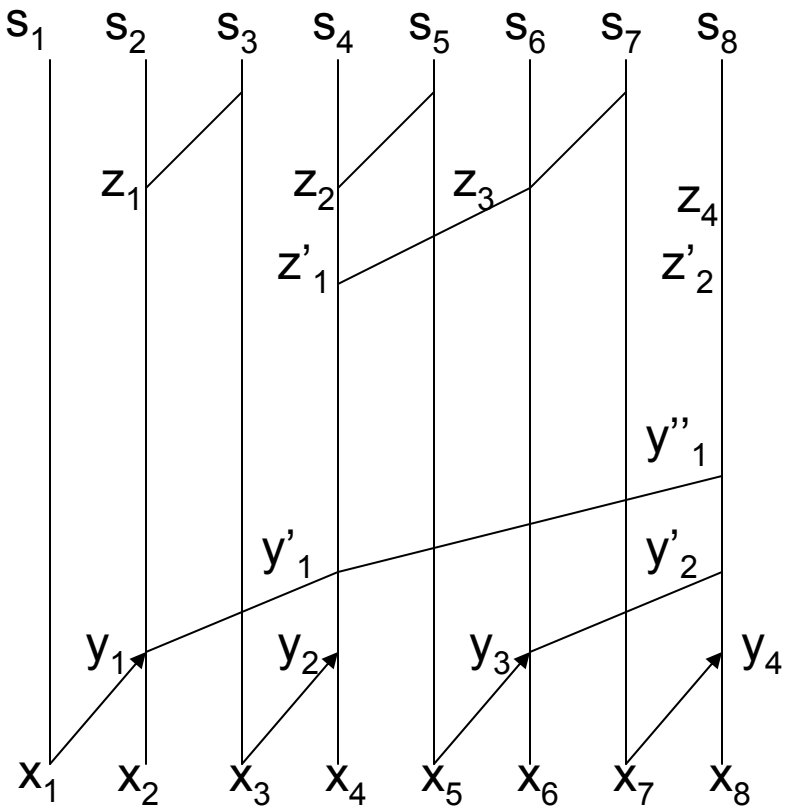
for $1 \le i \le n$ **pardo**

i even	$s_i := z_{i/2}$
i = 1	$s_1 := x_1$
i odd	$s_i := z_{(i-1)/2} * x_i$

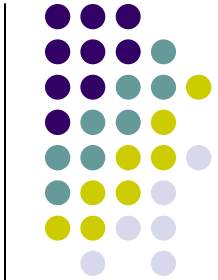
Prefix Computation



- Graphical illustration

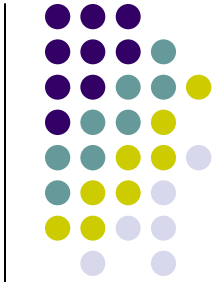


Prefix Sum



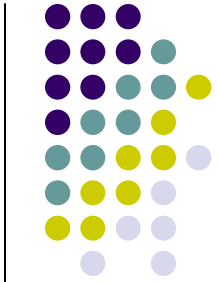
- We show algorithm does work W(n) = O(n) and uses time T(n) = O(log n).
- The work and time to compute $(z_1,\ldots,z_{n/2})$ from $(y_1,\ldots,y_{n/2})$ are W(n/2) and T(n/2). Additional work to compute prefix sum is *b*n for *b* >0. Additional time is *a* > 0.
  - W(n) = W(n/2) + *b*n
  - T(n) = T(n/2) + *a*
- These imply W(n) = O(n), T(n) = O(log n).

Examples of Prefix Computations



- Integer addition and multiplication
- Max and min
- Boolean AND and OR
- **copy_right**: a @ b = a

Prefix Computations



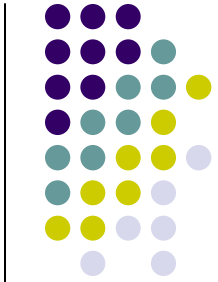
- **Segmented prefix computation**
  - Provide two n-tuple vectors:
    - Elements $(x_1,\ldots,x_n)$ from S and $(f_1,\ldots,f_n)$, f_j in B = {0,1}.
  - Output: $y_i = x_i$ if $f_j = 1$ **else** $y_i = x_i * y_{i-1}$.
  - Alternate input: $((x_1,y_1),\ldots, (x_n,y_n))$
  - New operation: & : $(S\times B)^2 \rightarrow S$
  - & is associative, i.e.
    - $(x_1,y_1)\&((x_2,y_2)\&(x_3,y_3)) = ((x_1,y_1)\&(x_2,y_3))\&(x_3,y_3)$

Finding Roots of a Tree in a Forest

- **Pointer doubling** in a linked list: replace the pointer from node *i* to its successor P(i) by a pointer from the successor, namely P(P(i)).
- Given a forest of rooted trees, find roots {S(i)}. A root r has S(r) = r.

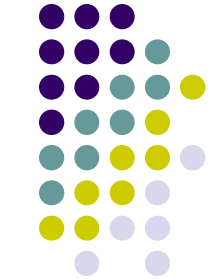
```
begin
for 1 ≤ i ≤ n pardo
        S(i) := P(i)
        while S(i) ≠ S(S(i) do
                S(i) := S(S(i))
```

Parallel Merging



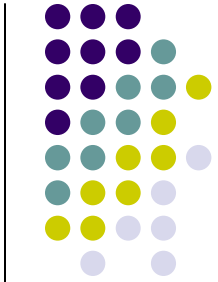
- Two sorted sequences of length n can be merged in O(log n) time on O(n) operations.
- **Goal:** Compute *rank*(A:B), the rank elements of A in B, two sets of distinct elements.
- **Approach:** rank individual elements of A in B. Rank x in A using *binary search* of x in A.

Parallel Partitioning on CREW



- Given sorted arrays of distinct elements A = $(a_1,\ldots,b_n)$ and B = $(b_1,\ldots,b_m)$, where $\log_2 m$ and $k(m) = m/\log_2 m$ are integers, partition B into k segments of length $\log_2 m$, $B_1, \ldots, B_k$.
- Partition A into k(m) contiguous segments $A_1, \ldots, A_{k(m)}$ such that elements in A_i and B_i are less than all elements in A_{i+1} and B_{i+1} and greater than all elements in A_{i-1} and B_{i-1}.
- Procedure: Compute $j(i) = \mathit{rank}(b_{i \log m} : A)$ and use to define $A_i = (a_{j(i)+1}, \ldots, a_{j(i+1)})$

Parallel Partitioning on CREW



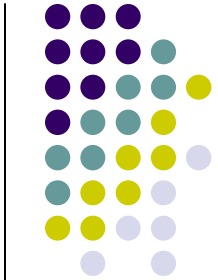
1. $j(0) := 0,\ j(k(m)) := n$
2. **for** $1 \le i \le k(m)-1$ **pardo**

   $j(i) := rank(b_{i \log m} : A)$ (use binary search)
3. **for** $0 \le i \le k(m)-1$ **pardo**

   $B_i = (b_{i \log m + 1}, \ldots, b_{(i+1) \log m})$

   $A_i = (a_{j(i)+1}, \ldots, a_{j(i+1)})$

Parallel Partitioning



Theorem Parallel partitioning of sorted lists $A = (a_1,\ldots,b_n)$ and $B = (b_1,\ldots,b_m)$ takes O(log n) time and O(n+m) operations.

Proof Step 1. takes O(1) sequential time.

Step 2. takes O(log n) parallel time. It uses O((log n) x (m/log m)) = O(n+m) operations.

Step 3. takes O(1) parallel time and uses linear number of operations.

Merging

- Merging of the segments A_i and B_i will produce sorted list. However, length of A_i may be large. To reduce running time, in parallel partition the pairs (B_i,A_i) so that segments have length at most $\log_2 n$.
- Merging of two sorted lists of length n on CREW takes time O(log n) and O(n) operations.
- Merging time can be reduced to O(log log n) on CREW by clever using of ranking.

Sorting

- Merge sort on lists of length 1, 2, 4, 8, etc. can be done in time $O(\log^2 n)$ using the partitioning algorithm mentioned above. It can be improved to O(log n log log n) on CREW.

Simulating CRCW on EREW PRAM

- The CRCW PRAM can be simulated on the EREW PRAM in time that is larger by a factor of the time to sort a list of p items, as we show.
- Consider a reading cycle. Processor p_j puts into location M_j value $(\mathbf{a}_j,j)$ indicating it wishes to read from location $\mathbf{a}_j$. Sort pairs. p_i reads location M_i on first step and loc. i-1 on second. If they hold different addresses, p_i reads from address given in M_i and puts it into M_i.

Simulating CRCW on EREW PRAM

- Using a segmented *copy-right* prefix computation (see book), copy value at location **a** to all locations M_k that request it. This step takes O(*log* p) time for p-processor PRAM. Sorting takes more time.
- To write, repeat except let p_j store value **v** to be written in M_j and let first (arbitrary) value be written.