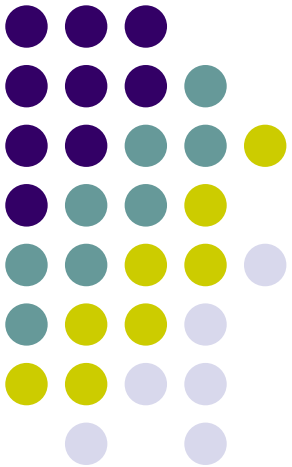
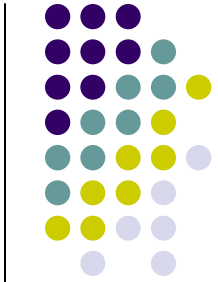
CS256
Applied Theory of Computation

Parallel Computation II

John E Savage

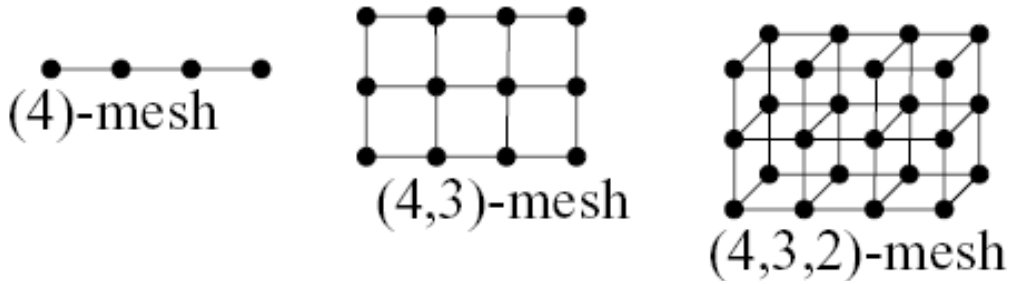


Overview



- Mesh-based architectures
- Hypercubes
- Embedding meshes in hypercubes
- Normal algorithms on hypercubes
- Summing and broadcasting on hypercubes
- Cyclic shift on hypercubes
- Cube-connected cycles – an architecture to efficiently embed hypercubes in the plane

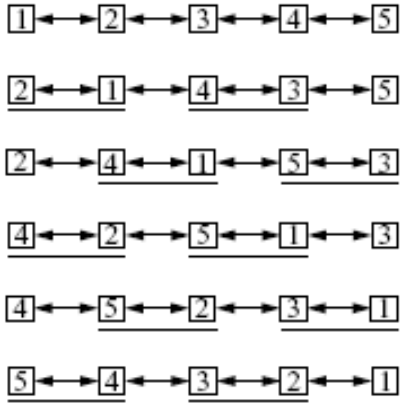
Mesh-Based Machines



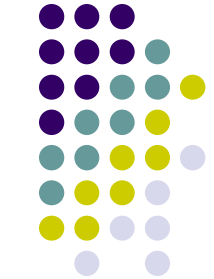
- d-tuple (n_1, n_2, ..., n_d) characterizes a d-dimensional mesh. n_i is number of vertices in the ith dimension. A (n_1, n_2, ..., n_d)-mesh is two (n_1, n_2, ..., n_{d-1})-meshes with an edge between corresponding vertices.
- It has $n_1 n_2 ... n_d$ vertices. The 1-D mesh has n_1-1 edges; the 2-D mesh has (n_1-1)(n_2-1) edges; and the d-D mesh has (n_1-1)(n_2-1)... (n_d-1) edges.

Sorting on a 1D Array

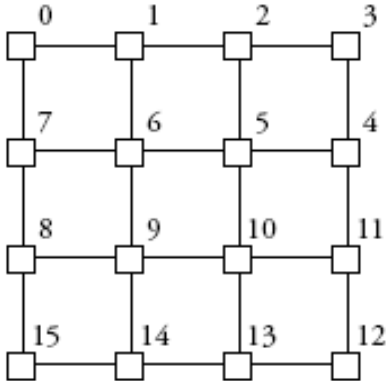
- **Bubble sort** compares adjacent elements from right to left and "bubbles up" the largest elements. Its running time is $O(n^2)$.
- **Odd-even transposition sort** compares odd with even elements and then even with odd elements n-1 times.
- Correctness shown using 0-1 principle.



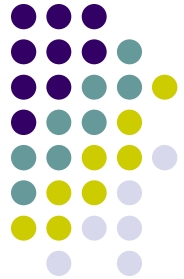
Embedding 1D Arrays in 2D Arrays



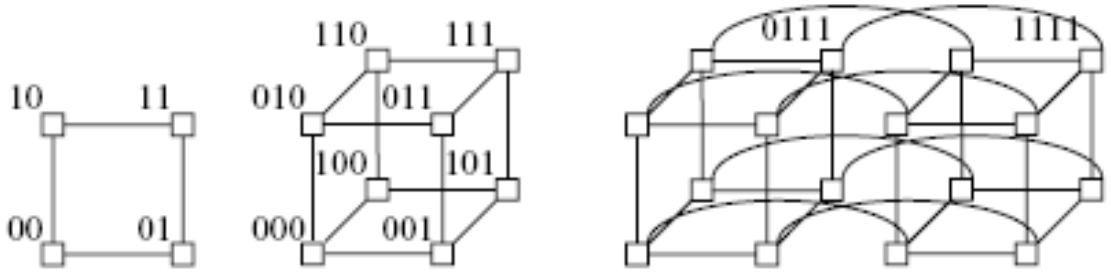
- Snake-row ordering of 1D array in 2D
  - Running time of 1D algorithm on 2D array unchanged.
  - Implementation of one step of 2D algorithm on 1D array in six steps:
    - Compute locally
    - Exchange data with column neighbors
    - Even (odd) rows send (receive) data from odd (even) rows.
  - Simulation of T-step n×n computation on 1D array takes at most (8n-1)T steps.



Hypercube-Based Machines



- A d-dimensional hypercube has 2^d vertices labeled by binary d-tuples. Edges between vertices whose d-tuples differ in one position.
- A d-D hypercube has (d/2) 2^d edges. It can be formed by adding edges between corres-ponding vertices of two (d-1)-D hypercubes.



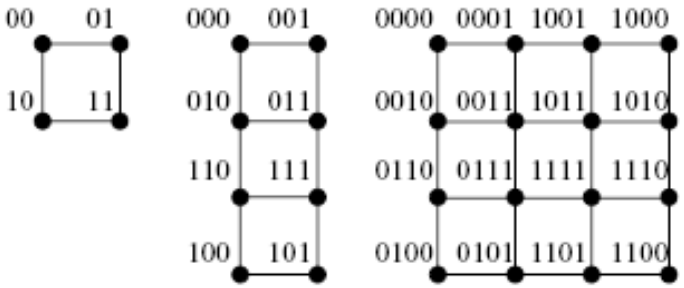
Graph Embeddings

- Graph G is **embedded** in graph G' by mapping each vertex of G with a unique vertex of G' and mapping edges of G into non-intersecting paths in G'.

- An embedding has a **dilation** of λ if the longest path into which an edge of the original graph is mapped is λ.

Embedding Meshes in Hypercubes

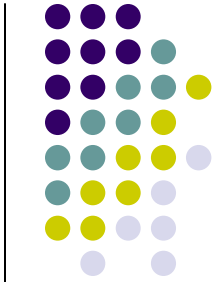
- Embeddings of meshes into 1D, 2D, and 3D hypercubes:



- How is this embedding done?

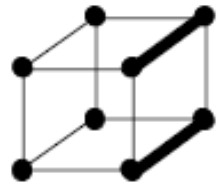
- In each embedding dilation $\lambda = 1$.

Embedding Meshes in Hypercubes



Theorem If d is even (odd), a $(2^{d/2}, 2^{d/2})$-mesh (a $(2^{(d+1)/2}, 2^{(d-1)/2})$-mesh) can be embedded in a d-D hypercube with dilation $\lambda=1$.

Proof (*Basis*) The (2,2)-mesh & 2-D hypercube are the same. The (4,2)-mesh can be embedded into the 3-D hypercube by cutting two edges on one face.



(*Induction Hypothesis*) Assume such embeddings exist for d < N. We show they exist for d=N.

(*Inductive Step*) d odd & even.

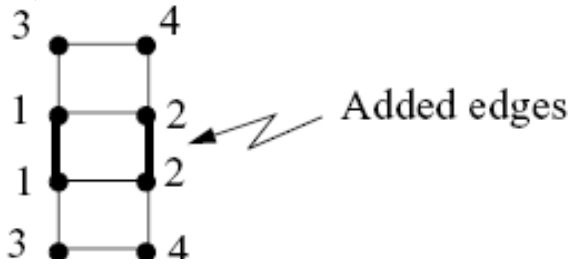
Embedding Meshes in Hypercubes

d odd

Let k=(d-1)/2 or d = 2k+1. The $(2^{k+1},2^k)$-mesh consists of two $(2^k,2^k)$-meshes. By hypothesis, each has embedding in (d-1)-D hypercube. Map to the (2k+1)-D cube as follows:

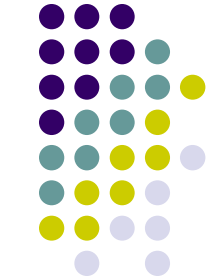
a) Map each $(2^k,2^k)$-mesh to a 2^{2k}-D cube.

b) Place both meshes on the plane, one above other. Flip the second mesh.



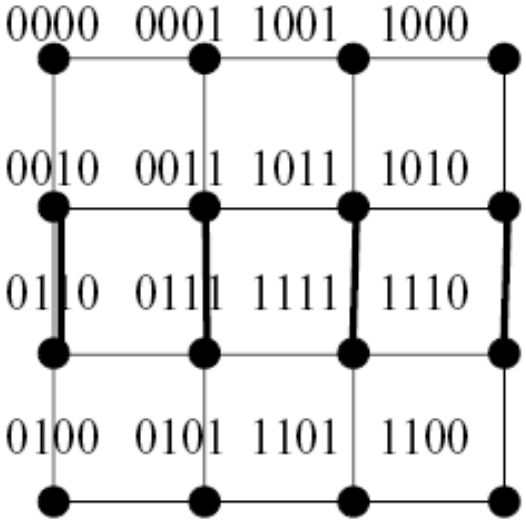
c) Add edges between the two faces. These edges are in the 2^{2k+1}-D cube containing the mesh, 2k+1 = d.

Embedding Meshes in Hypercubes



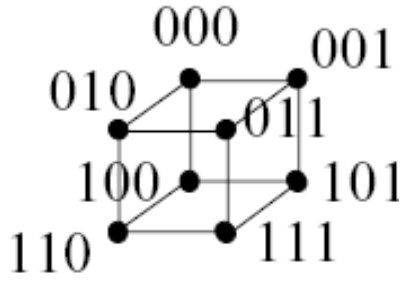
d even

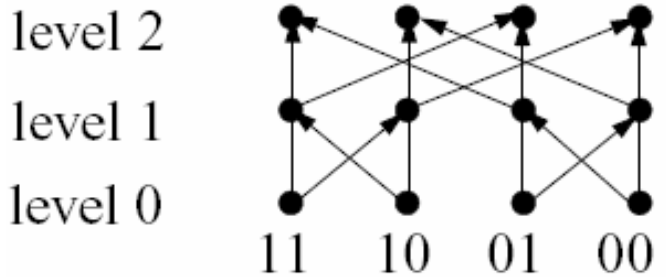
Let k =d/2. The $(2^k, 2^k)$-mesh can be represented as two $(2^k, 2^{k-1})$-meshes. Each can be embedded into (2k-1)-D hypercube as shown. These two resultant meshes are then laid out along their long sides and edges added so it can be embedded in hypercube.



Normal Algorithms

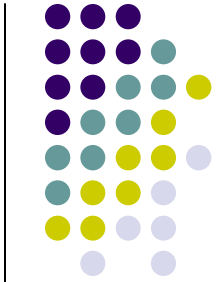
- A **normal algorithm** on a hypercube is one in which data moves synchronously across one dimension at a time.
- Many algorithms, such as FFT, are normal.





- Input data resides in vertices at level 0. To do computations at level 1, exchanges are done with neighbors differing on least significant bit. Computations at level 2 are done after exchanges between neighbors differing on most significant bit/

Normal Algorithms

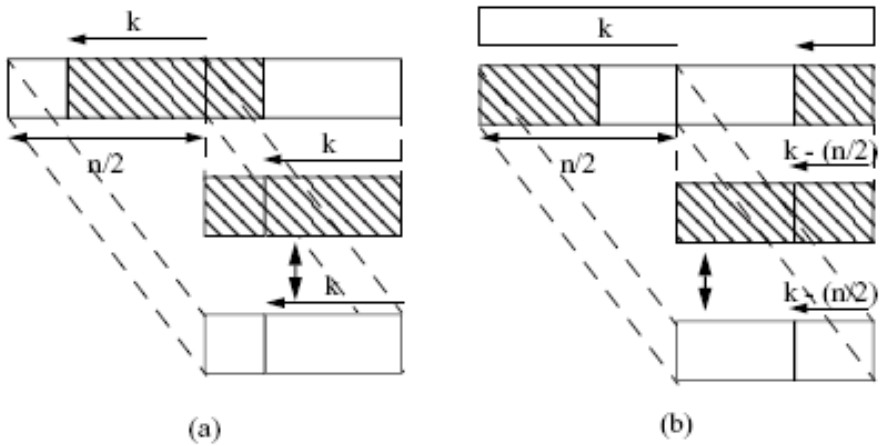


- **Ascending** normal algorithms make exchanges across dimensions 0, 1, 2, ...
- **Descending** normal algorithms make exchanges across descending dimensions.
  - Ex: FFT is an ascending normal algorithm
- Batcher's bitonic sort algorithm (see book – note correction) can also be realized as a normal algorithm on a hypercube.

Summing & Broadcasting on Hypercubes

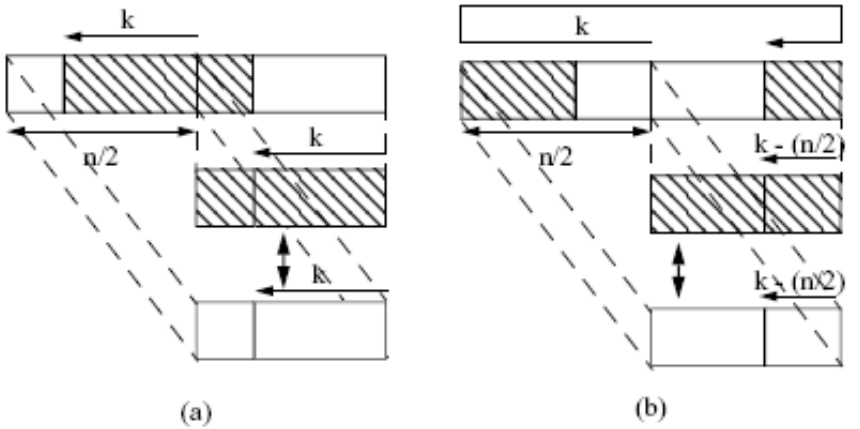
- **Summing:** A datum is placed on each vertex of a d-dimensional hypercube.
- On 1st cycle, for each tuple $(a_{(d-1)}, ..., a_0)$, vertex $(a_{(d-1)}, ..., a_1, \mathbf{1})$ sends it data to vertex $(a_{(d-1)}, ...,a_1,0)$ where it is added.
- On 2nd cycle vertex $(a_{(d-1)}, ..., a_2, \mathbf{1}, \mathbf{0})$ sends it data to vertex $(a_{(d-1)}, ..., a_2, \mathbf{0}, \mathbf{0})$ where it is added.
- Repeat until all data is summed at (0,0,..., 0)
- This is an ascending normal alg. Data moves across dimensions in ascending order.
- **Broadcast** by reversing the data transmissions.

Cyclic Shift on Hypercubes



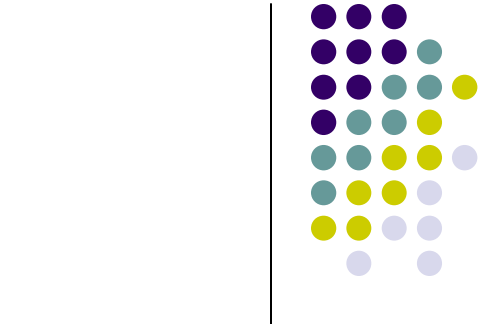
- Data to be shifted are stored on $n = 2^d$ nodes of a d-D hypercube. Order nodes, e.g. 000, 001, 010, … .
- To cyclic shift by k places when $k \leq n/2$, first shift half words by k places then swap first k positions across highest dimension, as shown. If $n = 2^d$, do this work recursively on a d-D hypercube.

Cyclic Shift on Hypercubes

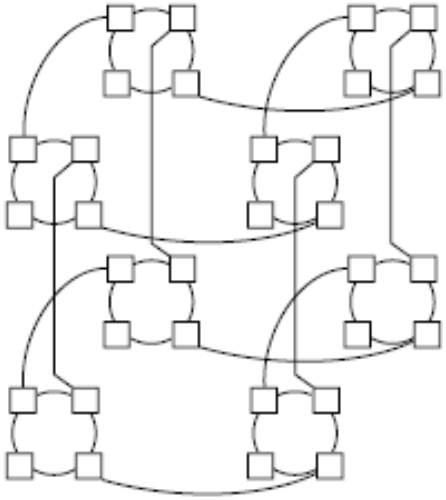


- To cyclic shift by k places when k > n/2, cyclic shift half words by k-(n/2) places and then swap top positions n-k positions, as shown. If $n = 2^d$, do this work recursively on a d-D hypercube.

Cube-Connected Cycles



- Each vertex of d-D hypercube has degree d. To reduce the degree, replace each vertex by a cycle of ≥ d vertices. Assign one vertex on each corner to each of its d edges.
- Connect the vertex assigned to a particular edge to the vertex on the opposite corner assigned to that edge. In above example, each cycle has 4 vertices, although 3 suffice.



Cube-Connected Cycles

- To swap across dimensions, move data around cycles replacing corners.

- The CCC is useful because ascending and descending normal algorithms can be implemented with only a small loss in time but the CCC can be laid out on the plane much more efficiently than the hypercube. (See book for details.)