CS256 Applied Theory of Computation

Memory Hierarchy Tradeoffs III

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Overview

- Application of the Hong-Kung Bound to the Fast Fourier Transform

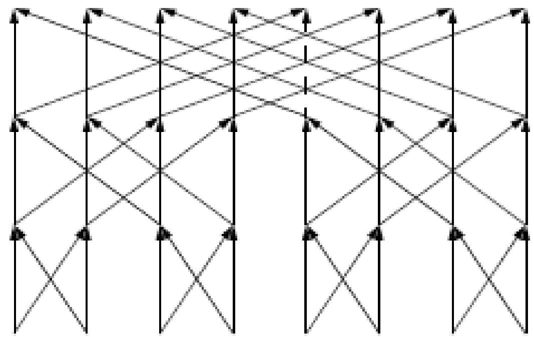
Fast Fourier Transform

Definition The **S-span** of DAG G, ρ(S,G), is the maximum number of vertices of G that can be pebbled with S red pebbles in red pebble game maximized over all initial placements of S red pebbles. (Initialization rule is disallowed.)

Theorem For every pebbling P of G = (V,E) in the red-blue pebble game with S red pebbles, the I/O time used, $T_2^{(2)}(S,G,P)$ satisfies

$$\left\lceil T_2^{(2)}(S,G,P)/S \right\rceil \rho(2S,G) \geq |V| - |In(G)|$$

Fast Fourier Transform (FFT)



- We derive matching upper and lower bounds on I/O complexity and number of computation steps to pebble the FFT graph $F^{(d)}$ on $n = 2^d$ inputs with S red pebbles.

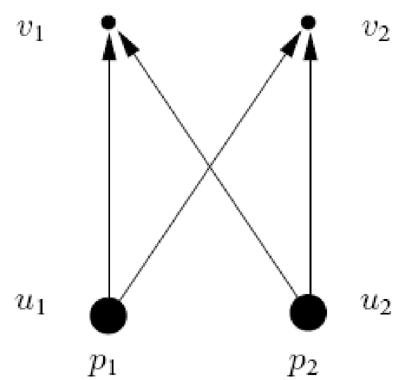
Fast Fourier Transform

Lemma The S-span of the FFT graph $F^{(d)}$ on $n = 2^d$ inputs satisfies $\rho(S,F^{(d)}) \leq 2S \log_2 S$ when $S \leq n$.

Proof Let $\{p_1, ..., p_S\}$ be pebbles used in game. We use num(p_i) (= 0 initially) to over bound the S-span. $F^{(d)}$ contains many 2-input FFT (butterfly) graphs.

Fast Fourier Transform

If v_1 *is* about to be pebbled, p_1 and p_2 must be on u_1 and u_2. Upper bound on S-span assumes v_2 also pebbled. That is, we advance pebbles to v_1 and v_2 (even though this violates the rules). If num(p_1) = num(p_2), increase each by 1. If not, increase smaller by 1. Because there are two pebble placements each time num(p_i) increases, $r(S,F^{(d)}) \le 2(\text{num}(p_1) + ... + \text{num}(p_S))$. We show that $\text{num}(p_i) \le \log_2 S$.



Fast Fourier Transform

Proof (cont.) Let p_i reside on v_i. Let N(i) be the number of initially pebbled predecessors of v_i. By induction we show that $N(i) \geq 2^a$, a = num(p_i).

Base case: num(p_i) = 1 corresponds to first move of p_i to a vertex with two initially pebbled predecessors.

Inductive hypothesis: $N(i) \geq 2^a$ for a = num(p_i) ≤ e-1. Consider first point in time at which num(p_i) = e. Let p_i move to v_i. At previous step p_i and second pebble p_j reside on predecessors u_1 and u_2 of v_i. Either a) num(p_i) = num(p_j) = e-1 or b) num(p_i) < num(p_j). In a) since u_1 and u_2 are roots of disjoint trees, the no. of predecessors of u_1 and u_2 carrying pebbles initially is at least $2(2^{e-1}) = 2^e$. In b), u_1 has at least 2^{e-1} predecessors with pebbles and u_2 has at least 2^e. Since at most S pebbles are on the graph initially, N(i) ≤ S, num(pi) ≤ $\log_2 S$ & $\rho(S,F^{(d)}) \leq 2S\log_2 S$ QED

Fast Fourier Transform

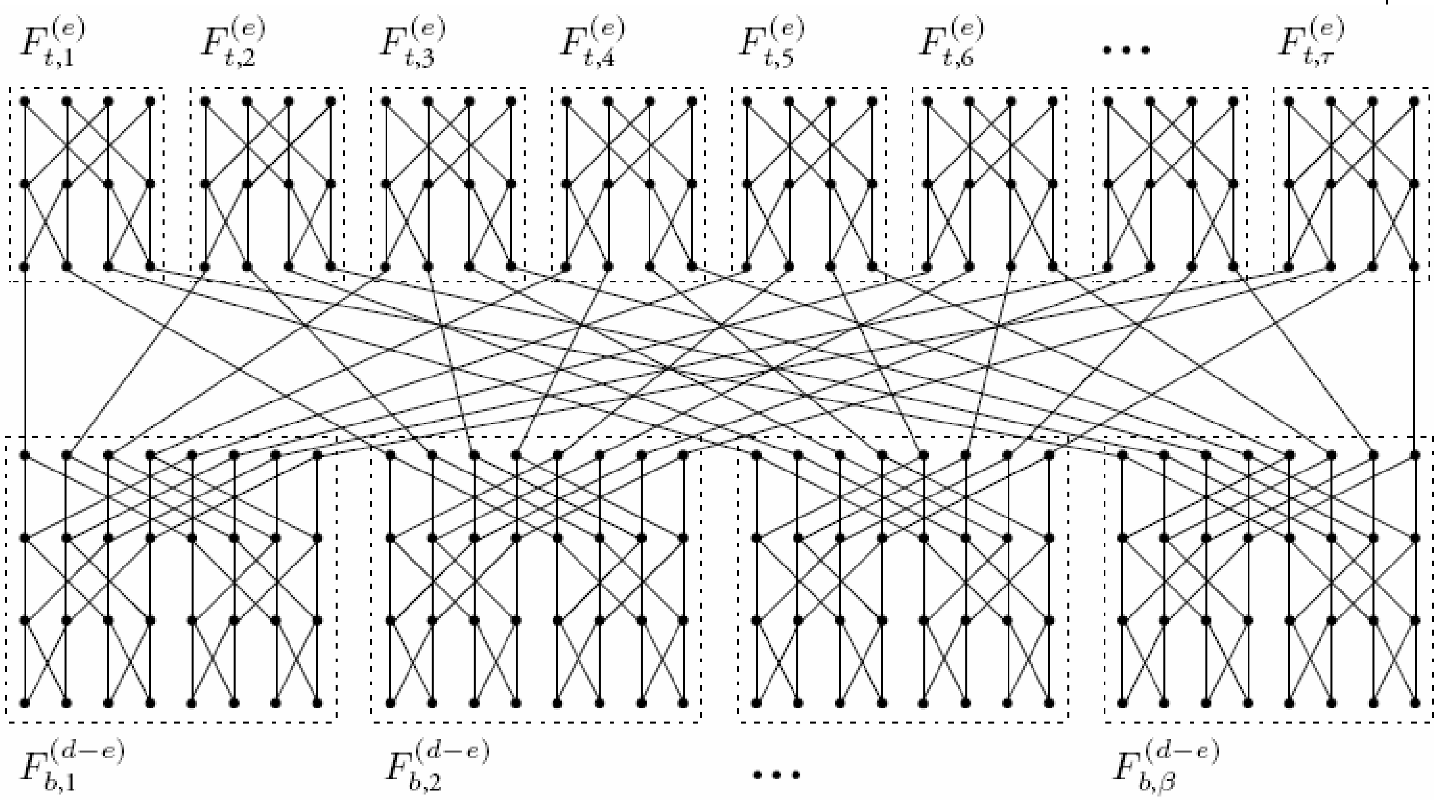


Figure 11.9 Decomposition of the FFT graph $F^{(d)}$ into $\beta = 2^e$ bottom FFT graphs $F^{(d-e)}$ and $\tau = 2^{d-e}$ top $F^{(e)}$. Edges between bottom and top sub-FFT graphs identify common vertices between the two.

Fast Fourier Transform

Theorem When $F^{(d)}$ pebbled in the red-blue pebble game with S ≥ 3, the following hold simultaneously:

$$T_1^{(2)}(S, F^{(d)}, \mathcal{P}) = \Theta(n \log n)$$

$$T_2^{(2)}(S, F^{(d)}, \mathcal{P}) = \Theta(n \log n / \log S)$$

Proof First lower bound is obvious. 2nd follows from S-span bound. Let β = 2^e*.* For upper bounds, consider decomposition into copies of $F^{(e)}$ on β inputs. Each vertex in $F^{(e)}$ is pebbled once with β +1 red pebbles. Blues are only used on inputs and outputs to $F^{(e)}$*.*

Fast Fourier Transform

Let $e = \lfloor \log_2 (S-1) \rfloor$ or $S \geq \beta + 1$ Then, we pebble all vertices in the top FFT graphs $F^{(e)}$ with blue pebbles only on inputs and outputs. We decompose each of the bottom FFT $F^{(d-e)}$ graphs into graphs $F^{(e)}$ at the top with $F^{(d-e)}$ graphs on the bottom and pebble the graphs $F^{(e)}$ in the same way. Thus, we pebble with blue pebbles the vertices at levels in $F^{(d)}$ separated by $\log_2 (S-1)$ levels. Since there are $(\log_2 n + 1)$ levels and n vertices per level, this gives the upper bound on $T_2^{(2)}(S, F^{(d)}, \mathcal{P})$. The upper bound on $T_1^{(2)}(S, F^{(d)}, \mathcal{P})$ follows because each vertex is pebbled once with a red pebble. QED