CS256
Applied Theory of Computation

Circuit Complexity IV

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Overview

- Slice Functions
- Pseudo-negation
- Monotone circuit size of slice functions
- Half-Clique Central Slice – an NP-complete problem

Slice Functions

- Some monotone functions have exponential circuit size over the monotone basis, doubtful that same methods of analysis can be extended to derive such bounds over the standard basis.
- We show that the monotone circuit size of monotone slice functions can provide a strong lower bound on the circuit size of such functions over the standard basis. Also, there are **NP**-complete languages whose characteristic functions are slice functions. Thus, if such functions can be shown to have super-polynomial monotone circuit size, **P** ≠ **NP**.

Slice Functions

- Let $|\mathbf{x}|$ be no. 1's in **x**. A **slice function** s(**x**) has value 0 for $|\mathbf{x}| < k$ & value 1 for $|\mathbf{x}| > k$ for some k.
- For $f : B^n \to B$, $f^{[k]}$ is the **k th slice function** if it is 0 for $|\mathbf{x}| < k$, 1 for $|\mathbf{x}| > k$, and equal to f otherwise.
- Note that slice functions are monotone!

Slice Functions

Lemma For $f : B^n \rightarrow B$ over the standard basis

$$C(f) = C(f^{[0]}, f^{[1]}, ..., f^{[n]}) + O(n)$$

Proof Given $f^{[0]}, f^{[1]}, ..., f^{[n]}$ we construct a circuit for f from the multiplexer function and a circuit to count the number of 1s in the input. Supply output of the counting circuit to input of the multiplexer along with the outputs of the circuits for $f^{[0]}, f^{[1]}, ..., f^{[n]}$.

Slice Functions

- **The following result says that for every Boolean function there is some slice of it that has a circuit size close to the circuit size of the entire function when its circuit size is large.**

  Theorem For $f : B^n \rightarrow B$ over the standard basis **there exists k** such that

$$C(f)/n - O(1) \leq C(f^{[k]}) \leq C(f) + O(n)$$

Slice Functions

Theorem For $f : B^n \rightarrow B$ over the standard basis **there exists k** such that

$$C(f)/n - O(1) \leq C(f^{[k]}) \leq C(f) + O(n)$$

Proof Because $C(f^{[0]}, f^{[1]}, ..., f^{[n]}) \leq \Sigma\, C(f^{[i]})$, there must be $i = k$ for which $C(f^{[i]})$ is largest. The lower bound follows from this. The upper bound follows from $f^{[k]}(x) = (\tau^{(n)}_k(x) \wedge f(x)) \vee \tau^{(n)}_{k+1}(x)$ where $\tau^{(n)}_k$ is the threshold function with threshold k.

Pseudo-Negation

Definition A **pseudo-negation** for a variable x_i in a function f is a function h such that replacing each instance of x_i by h does not change the value of f.

The **punctured threshold function** $\tau^{(n)}_{k,i}$ is $\tau^{(n-1)}_{k}$ applied to all variables except x_i.

Since $\tau^{(n)}_{k,i}$ can be realized by a binary sorter, $C_{mon}(\tau^{(n)}_{k,i})$ is O(n log n). Thus, all n punctured threshold functions on n variables can be realized by $O(n^2 \log n)$ gates over the monotone basis. This bound can be improved to $O(n \log^2 n)$.

Monotone Circuit Size of Slice Functions

- The following theorem shows that if the monotone circuit size of a **slice function** is large, its standard circuit size is also large.

  Theorem Let $f : B^n \rightarrow B$ be a slice function and $C_{mon}(f)$ and $C(f)$ be its circuit size over the monotone and standard bases. Then,

$$C(f) \leq C_{mon}(f) \leq 2C(f) + O(n \log^2 n)$$

Monotone Circuit Size of Slice Functions

Proof Convert an optimal circuit for f to dual-rail logic. This at most doubles the number of gates and has as inputs x_i and $\bar{x}_i$.

Let k = threshold of f. f = 0 (1) if $|\mathbf{x}| < k$ ($> k$).

$\tau^{(n)}_{k,i} = 0$ (1) when $|\mathbf{x}| < k$ ($> k$).

When $|\mathbf{x}| = k$, $\tau^{(n)}_{k,i} = 0$ (1) if $x_i = 1$ (0).

Monotone Circuit Size of Slice Functions

Replace $\bar{x}_i$ by $\tau^{(n)}_{k,i}$. When $|\mathbf{x}| \leq k$, $f = 0$ whether $x_i = 0$ or 1. Replacing $\bar{x}_i$ by $\tau^{(n)}_{k,i} = 0$ doesn’t change f.

When $|\mathbf{x}| \geq k$, $f = 1$ whether $x_i = 0$ or 1. Replacing $\bar{x}_i$ by $\tau^{(n)}_{k,i} = 1$ doesn’t change f.

Finally, when $|\mathbf{x}| = k$, $\tau^{(n)}_{k,i}$ behaves like $\bar{x}_i$.

Central Slice

Definition The **central slice** of $f : B^n \rightarrow B$, is $f^{[k]}$ for $k = \lceil n/2 \rceil$.

Definition The **central clique function** $f_{clique,n/2}^{(n)}$ on $n(n-1)/2$ inputs (denoting presence or absence of edges in a graph on n vertices) has value 1 if the graph has a clique on $\lceil n/2 \rceil$ vertices and 0 otherwise

Let $e(k) = k(k-1)/2$ be number of edges in a k-clique. The central slice of $f_{clique,n/2}^{(n)}$ is called the **half clique central slice function** $f^{(n)}_{clique\ slice}$. It has value 1 if the graph denoted by inputs has a clique on $\lceil n/2 \rceil$ inputs or it has more than $e(n/2)$ edges.

Central Slice

HALF CLIQUE CENTRAL SLICE

Input: Undirected graph G = (V,E), |V| even

Answer: "Yes" if G has a clique on |V|/2 vertices or it has at least e(|V|/2) edges.

Half Clique Central Slice

- The following theorem demonstrates that the half clique central slice function has a monotone circuit size at least as large as every slice function of the central clique function. Combined with earlier results, this demonstrates that the central slice of the half-clique function on n/2 vertices has polynomial-size circuits if and only if the half-clique function does.

  Theorem HALF CLIQUE CENTRAL SLICE is **NP**-complete. Furthermore, for all $1 < k < n$,

$$C_{mon}((f_{clique,n/2}^{(n)})^{[k]}) \leq C_{mon}(f^{(n)}_{clique\ slice})$$

Half Clique Central Slice

Proof Reduce from instance G = (V,E) with n vertices, n even, of HALF-CLIQUE to instance G' = (V',E') of HALF CLIQUE CENTRAL SLICE that has n'= 5n vertices such that G has a clique on n/2 vertices or more than k edges if and only if G' has a $\lceil 5n/2 \rceil$ - clique or more than $\lceil e(\lceil 5n/2 \rceil)/2 \rceil$ edges.

Half Clique Central Slice

Let $V = \{v_1, ..., v_n\}$. Build G' by adding vertices $R = \{r_1, ..., r_{2n}\}$ and $S = \{s_1, ..., s_{2n}\}$. Represent edges with variables $\{y_{i,j} \mid 1 \leq i, j \leq 5n\}$.

Fix the $e(4n)+4n^2$ edges F between vertices within R and S and between R, S and V as follows.

Set $y_{i,j}$ so that no edge exists between r_i and s_i $1 \leq i \leq 2n$, R is a clique, and each vertex in V is connected to each vertex in R. This fixes $4n^2+n$ of the edges F leaving $8n^2 - 3n$ edges to be fixed.

Half Clique Central Slice

Let $r = \lceil e(\lceil 5n/2 \rceil)/2 \rceil - (4n^2+n)$. Clearly, $r \leq 8n^2 - 3n$. Fix the remaining vertices so that F has r-p edges.

If G has an (n/2)-clique, G' has a (5n/2)-clique. On the other hand, if G' has a (5n/2)-clique, G must have a (n/2)-clique because no clique of this size can include S.

Also, G has more than p edges iff G' has more than $\lceil e(\lceil 5n/2 \rceil)/2 \rceil - p$ edges.

Half Clique Central Slice

- The membership of a graph G in HALF-CLIQUE can be determined by determining membership of G’ in HALF-CLIQUE CENTRAL SLICE. This the latter problem is **NP**-hard. Because it is in **NP**, it is **NP**-complete.