Complexity Classes VIII

Stronger Approximation Bounds

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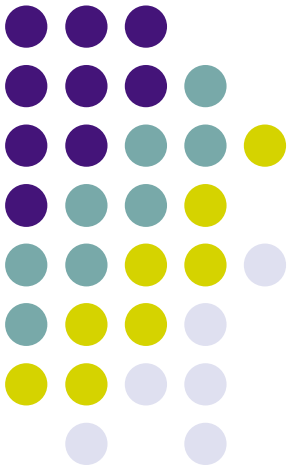
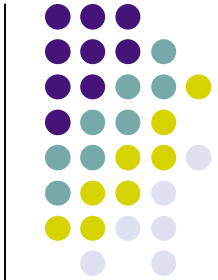
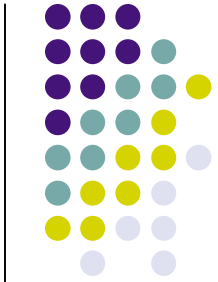


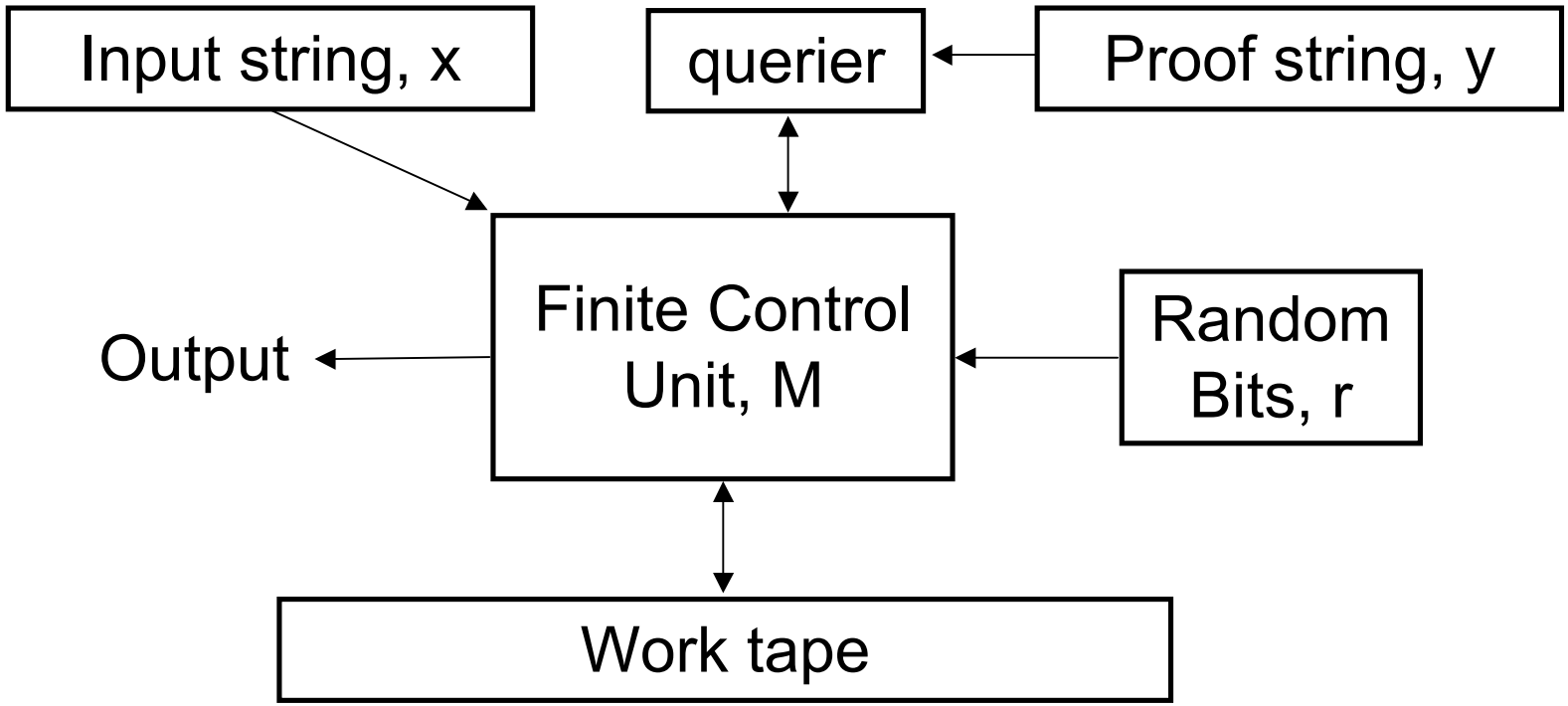
Review of PCP



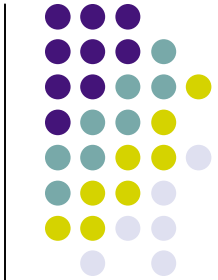
- **$PCP_{c,\,s}[q(n), r(n)]$** is the class of languages that can be recognized with by some Turing machine (which we call a "verifier") with soundness s (or less) and completeness c (or more) using O(r(n)) random bits and O(q(n)) queries to a proof.
  - Completeness c means that there exists a proof such that strings in L are accepted with probability c.
  - Soundness s means that for all proofs the TM accepts strings not in L with probability s.

PCP Verifiers



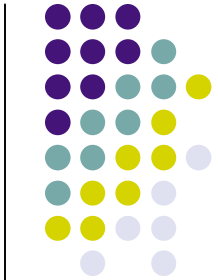


The PCP Theorem



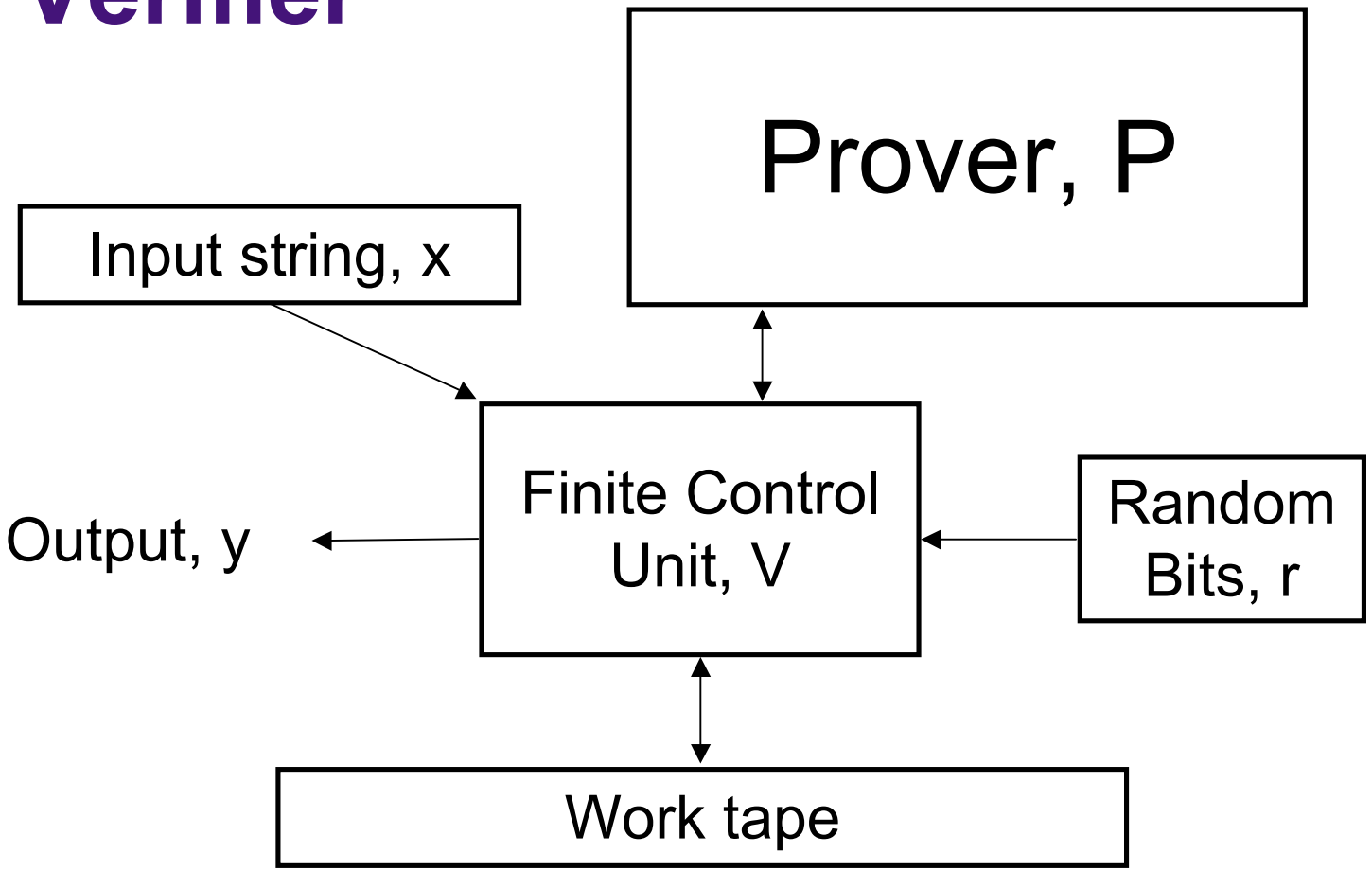
- PCP Theorem: NP = $PCP_{1,\ 1/2}$ [1, log(n)].
  - which implies NP = $PCP_{1,\ 1/n}$ [log(n), log(n)].
- By representing the behavior of a verifier as an instance of 3SAT, we saw that a PTAS for 3SAT implied P = NP.
- By representing the behavior of a verifier as an instance of clique, we were able to show a constant factor approximation for CLIQUE implied P = NP.
- In both cases, our results were functions of q, the number of queries required to verify some NP-complete language.

IP



- IP is the class of languages “recognized” by a polynomial rounds of randomized interaction:
  - $q_1 = V(x, r_1)$
  - $a_1 = P(x, q_1)$
  - . . .
  - $a_{poly(|x|)} = P(x, q_1, a_1 , . . . , q_{k-1}, a_{k-1}, q_k)$
  - $y = V(x, q_1, a_1 , . . . , q_{k-1}, a_{k-1}, q_k, r_k)$
- V is PTIME, P can safely be restricted to PSPACE.
  - Since P takes a random input, r, languages are recognized with completeness and soundness parameters.
- We saw last time that IP = PSPACE.

IP Verifier

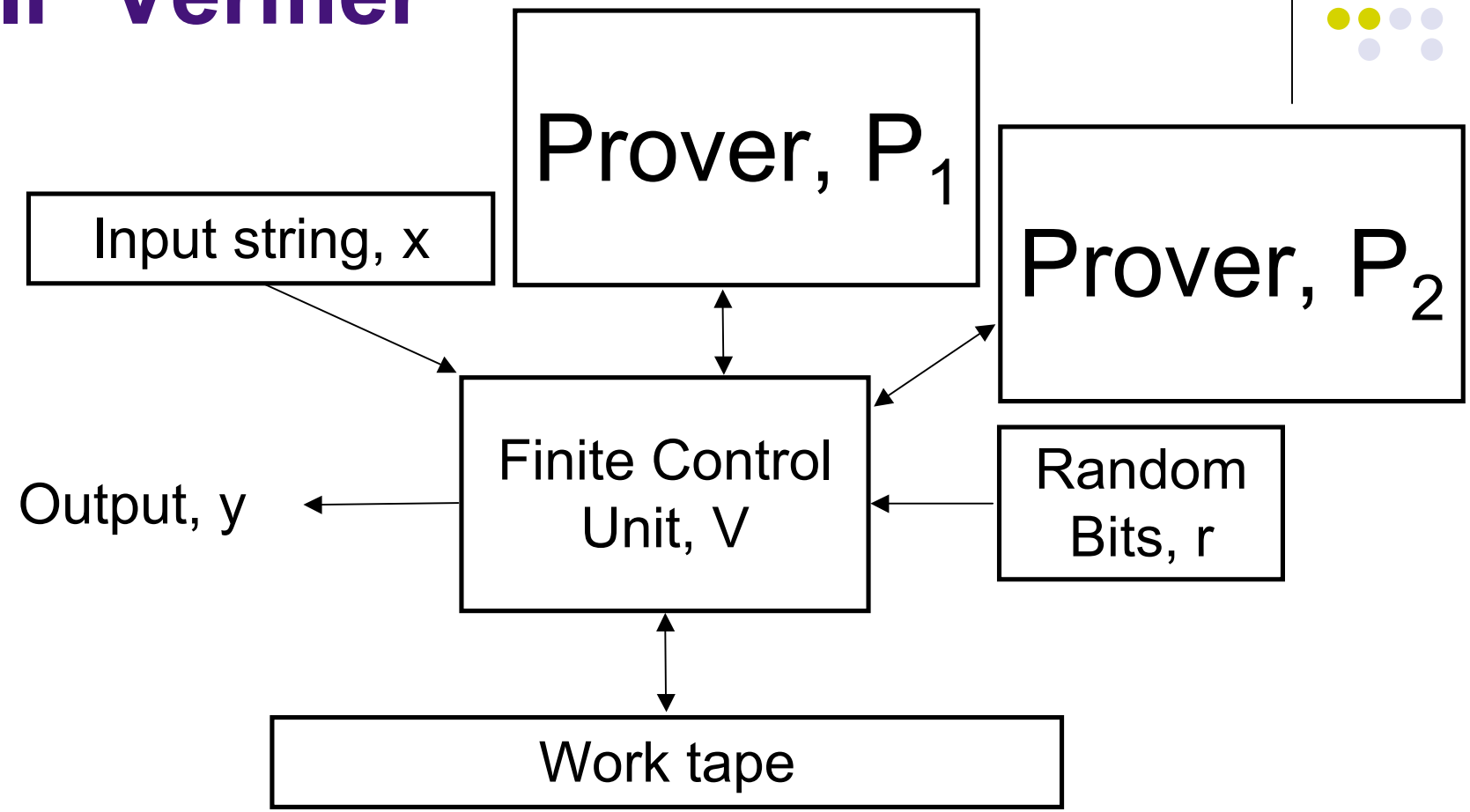

Prover, P
Input string, x
Finite Control Unit, V
Output, y
Random Bits, r
Work tape

MIP

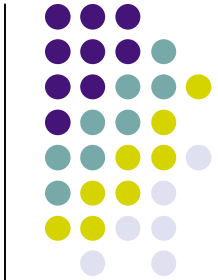
- MIP is defined like IP, except with multiple provers. The provers can be pitted against each other.
  - It turns out MIP = NEXPTIME.
  - Just two provers and one round of interaction suffice.
- In either IP or MIP, multiple repititions of the proof protocol can drive error rates exponentially low.
- With MIP, we have an additional option. Ask many questions at once.
  - This is known as parallel repetition.

*See end of lecture 7 for citations.

MIP Verifier



PCP and MIP

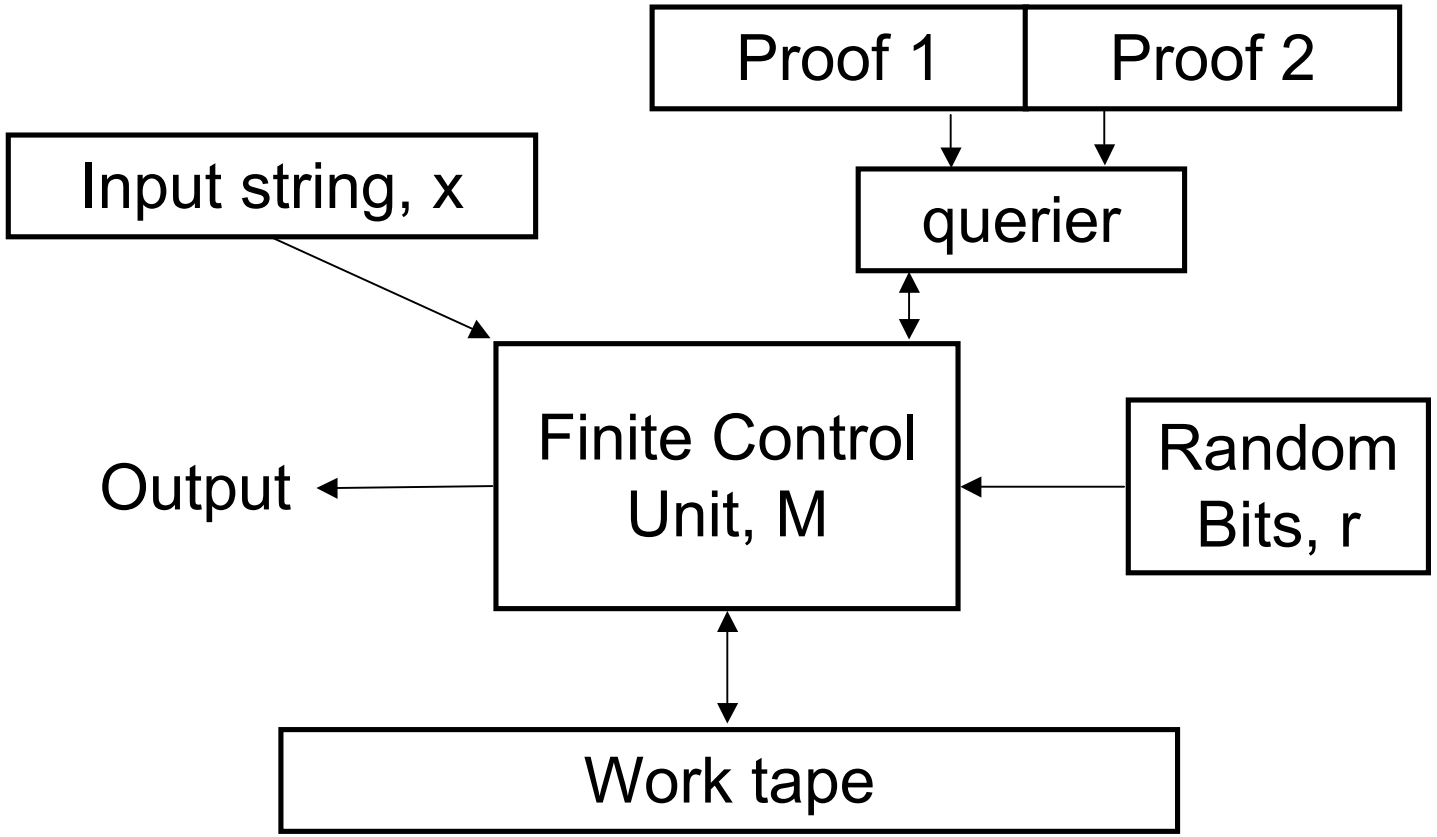


- MIP is incredibly powerful, but what if we keep questions short (O(log(n))?
- The PCP theorem implies NP-Complete problems can be reduced to gap instances of 3SAT.
- We have short a two prover one round protocol:
  - Ask P_1 the assignment to the three variables in a clauses
  - Ask P_2 the assignment to one of the variables
  - Check for disagreement.
- Multiple sequential repetitions increase soundness.
  - What about asking multiple questions at once? Yes!
  - This results from the parallel repetition theorem (Raz, 1998)

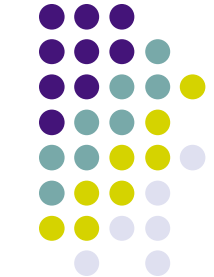
Parallel Repetition

- The parallel repetition theorem says that
  - Through t parallel repetitions, we can reduce the soundess from s to $(1 - s^a)^{bt}$, where a, b = o(1)
- A PCP can be considered a one round MIP.
  - Different parts of the proof represent each prover.
- This implies the result we proved:
  - $PCP_{1,\ 1/2}\ [1,\ \log(n)] = PCP_{1,\ 1/n}\ [\log(n),\ \log(n)]$.
- Since MIP = NEXPTIME, we also have:
  - $NEXP = PCP_{1,\ exp(-n)}[poly(n),\ poly(n)]$

MIP with PCP Verifier



A 2 query PCP (sort of)



- Now we have a two query PCP, in a larger alphabet.
- We know that NP-complete problems can be reduced to 3SAT via a gap introducing reduction.
- Using our 2 prover model
  - Ask the first prover the assignment to all variables in some set of k clauses.
  - Ask the second the assignment to some set of k variables, at least one from each clause.
  - The parallel repetitions theorem states that regardless of the original gap, there is some k such that our soundness is close to 0.

Reducing the Alphabet

- In the PCP theorem, we were able to reduce the alphabet from {1,…,m} to {0,1} using Walsh-Hadamard Codes.
  - If x is the binary representation of a variable ranging 1 to m, WH(x) lists all m different sums of x's bits.
  - WH(x) and WH(y) are 1/2-close if $x \neq y$.
  - If WH'(x) is ε-close to WH(x), we can use two queries to determine any entry in WH(x) with probability at least 2ε.
    - RECALL: $WH(x)_s = WH(x)_{s'} + WH(x)_{s'+s}$.
  - If WH'(x) is not ε-close to any codeword WH(x), it turns out we can detect this with probability at least min(ε, 1/2).
    - $\text{Prob}[WH'(x)_s = WH'(x)_{s'} + WH'(x)_{s'+s}] \leq \max(1 - \varepsilon, 1/2)$
- Now we use an even longer code…

The Long Code

- If x is a b = log(m) bit string, LONG(x) has 2^m entries, one for each function of x.
- A function of b bits is represented by a $2^b = m$ row truth table. There are 2^m possible functions.
- Again, LONG(x) and LONG(y) are 1/2-close if $x \neq y$.
- If LONG'(x) ε-close to LONG(x), we can use two queries to find any f(x) with probability at least 2ε.
  - $LONG'(x)_{f(x)} = LONG'(x)_{g(x)} + LONG'(x)_{f(x)+g(x)}$
- Again, if LONG'(x) is not close to a codeword, we can detect this with probability close to 1/2
  - TEST: $LONG'(x)_{f(x)} = LONG'(x)_{g(x)} + LONG'(x)_{f(x)+g(x)}$

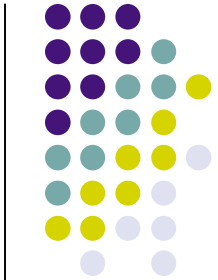
PCP using the Long Code

- Our 2 query PCP verifier asked two questions:
  - What are the values in k clauses?
  - What are the values of k variables?
- It then checks if the two answers, a_1 and a_2, are consistent:
  - For some function h, check $h(a_1) = a_2$.
- If we pick a second random boolean function, f, we can perform the check with probability 1/2.
  - For some random f, let $g(x) = f(h(x))$, check $g(a_1) = f(a_2)$.
  - The soundness of this test can't be guaranteed.
- This type of test uses the long codes of a_1 and a_2.

A Three Query PCP

- Assume the proof consists of LONG(a_1) and LONG(a_2) for every possible pair of queries.
- We check $g(a_1) = f(a_2)$ using a random function, g'(x)
  - $\text{LONG}(a_2)_{f(x)} = \text{LONG}(a_1)_{g'(x)} + \text{LONG}(a_1)_{g(x) + g'(x)}$
- This test would have perfect completeness, but it must be modified slightly.
  - When checking $\text{LONG}(a_2)_{f(x)}$, we sometimes check f's compliment.
  - The function indexed in the last term must have a small amount of "random noise" added to it.
- It leads to a 3 query PCP with completeness 1 - ε and soundness 1/2 + ε.
- For each randomly chosen pair of questions, the test is linear.

Hardness of 3SAT

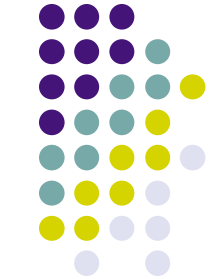


- The protocol shows NP = $PCP_{1-\varepsilon,\ 1/2+\varepsilon}$ [3, log(n)].
  - Language in NP can be recognized by a verifier M that makes 3 queries to a proof y given some random input r.
  - In other words, $M_{r,\ x}(y)$ is a function of 3 bits of y, plus it is linear.
- The DNF of each $M_{r,\ x}(y)$ involves 4 clauses of 3 variables each.
- If a string is not in the language, close to 1 out of 8 clauses is unsatisfied.
- Unless P = NP, the known 7/8-approximation of 3SAT is the best possible.

Completeness and Soundness

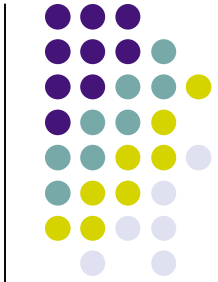
- Proving completeness is easy. If the answers are correctly encoded, we will accept w.h.p.
- Proving soundness is more difficult. To show it is difficult to cheat, Fourier Analysis is used.
- We prove the simpler soundess result we used in Step 3 of the PCP theorem.
  - If WH'(x) is not ε-close to any codeword WH(x), $\text{Prob}[WH'(x)_s = WH'(x)_{s'} + WH'(x)_{s'+s}] \leq \max(1 - \varepsilon, 1/2)$

Soundness of WH'(x)



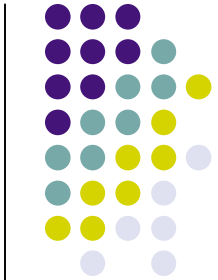
- For a binary string x in $\{0,1\}^b$, WH(x) lists all $2^b = m$ sums of x's bits.
  - Each sum is indexed by one of m binary b-tuples.
- For the purpose our analysis, we take binary to mean {-1, 1} (that is, 0 is replaced with -1).
  - Now WH(x) is all m products of x's bits.
  - Also, let dot(x, y) = $E[x_i y_i]$ (meaning the standard dot product by m)
- The Fourier basis of $\{0,1\}^m$ is f_α for each subset α of {1,…,b}, where $f_{\alpha,x} = \prod_{i \in \alpha} x_i$.
  - Each f_α corresponds to a linear functions of $\{0,1\}^b$.
- Notice that the basis is orthonormal, so every WH(x) can be represented as the sum of basis elements, $\sum WH_\alpha(x)$.
- Also dot(WH(x), WH(y)) = $\sum WH_\alpha(x)WH_\alpha(y)$.

More Soundness



- Using our new alphabet of {-1, 1}, we wish to show that:
  - If Prob[$WH'(x)_s = WH'(x)_{s'}WH'(x)_{s'\bullet s}$] ≥ 1/2 + ε, there $WH'(x)$ there is some basis element f_α such that dot($WH'(x)$, f_α) ≥ 2ε.
  - We call dot($WH'(x)$, f_α) fourier coefficient c_α of $WH'(x)$
  - s' is chosen at random and s'•s is pairwise multiplication.
- We can prove $E[WH'(x)_s WH'(x)_{s'}WH'(x)_{s'\bullet s}] \geq 2\varepsilon$
  - Write each codeword as sum of Fourier coefficients.
  - Simplify using $f_{\alpha, s'\bullet s} = f_{\alpha, s'}f_{\alpha,s}$
  - Use linearity of expectation
  - Simplify with orthonormality

Conclusion



- Using the PCP theorem and parallel repetition, we can construct a high soundness, high completeness PCP.
- Using the Long Code, we can ask long questions using only a few queries.
- Combining these, Hastad (2001) was able to show that NP = $PCP_{1-\varepsilon,\ 1/2+\varepsilon}$ [3, log(n)].
- This implied that a PTIME 7/8-approximation for 3SAT is as good possible unless P = NP.
- The approach applies to other problems as well.