Complexity Classes VII

Interactive Proofs

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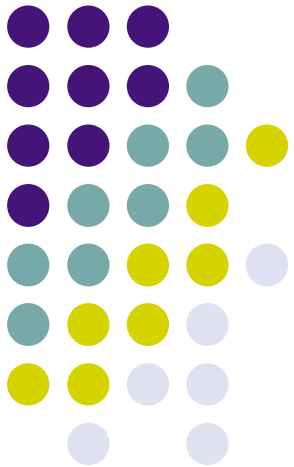
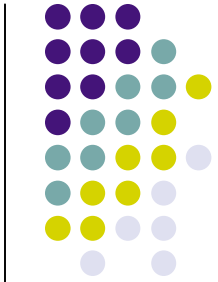
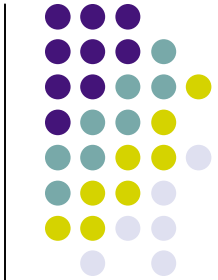


Verifiers and Provers



- NP: Languages that can be recognized in PTIME with access to a certificate or proof.
- PCP: Languages that can be recognized in PTIME with limited access to a proof. Here the verifier uses random bits.
- IP: Languages that can be recognized in PTIME with access to ``Prover'', a machine that answers a series of questions.
- MIP: Like IP, only with multiple provers.
  - Historically, IP and MIP came before PCP

What good is a prover?

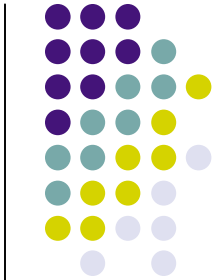


- Intuition: To prove a statement to you, I could write down a lengthy proof. Rather than verify a long proof, it's easier to adaptively ask me questions.
- Example: Graph Isomorphism:
  - To show G_1 and G_2 are not isomorphic, a verifier can repeatedly pick one at random, randomly permute it, and ask the prover if the new graph came from G_1 or G_2.
  - If the prover is ``smart enough'', and the graphs really aren't isomorphic, it will always answer correctly right.
  - If the two graphs are isomorphic, the prover will be unable to answer correctly more than half the time.

Modeling k-rounds of Deterministic Interaction

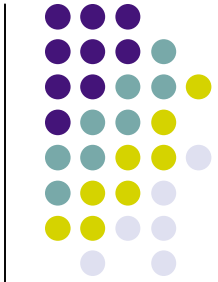
- Let x be a binary string, L be a language, and V and P be functions $\{0, 1\}^n \rightarrow \{0,1\}^m$, where n and m are polynomial in |x|.
- Consider the following k step interaction, V (but not necessarily P) is PTIME, and y indicates if x is in L:
  - $q_1 = V(x)$
  - $a_1 = P(x, q_1)$
  - . . .
  - $q_k = V(x, q_1, a_1 , . . . , q_{k-1}, a_{k-1})$
  - $a_k = P(x, q_1, a_1 , . . . , q_{k-1}, a_{k-1}, q_k)$
  - $y = V(x, q_1, a_1 , . . . , q_{k-1}, a_{k-1}, q_k)$

IP[k] without randomness



- What languages can be recognized by k steps of deterministic interaction?
  - Since the verifier is deterministic, P, given x, could simply write down all k answers in advance.
  - The verifier could then verify these are the answers the the questions it would have asked.
- What if we allow k to be polynomial in |x|?
  - If k is polynomial in |x|, the answers are polynomial in x, and the languages recognized are those in NP.

The Real Deal: IP

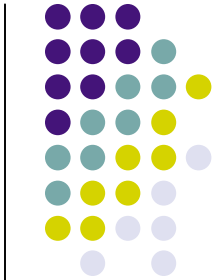


- Let $IP_{c,s}[k]$ be the languages recognized by k rounds of interaction, where V is now a random function.
  - V always gets a random sequence r as input, where |r| is polynomial in |x|.
  - Now ``recognized'' is given in terms of completeness, c, and soundness, s.
- Let IP be the languages recognized by k(x) rounds of interaction, where k(x) is any polynomial, c = 2/3 and s = 1/3.

More on the Definition of IP

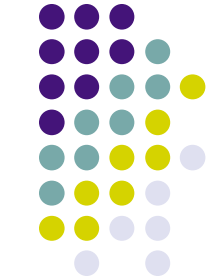
- It turns out IP is the same if c = 1
  - If s = 0, however, IP becomes NP.
- The power of P is unbounded, but in fact P can be restricted to functions in PSACE.
- We gave V access to a random string, why not P?
  - If a random verifier works, a deterministic verifier could just pick the answer most likely to succeed.
- Is it necessary that V's random bits be kept secret from P?
  - AM[k] denotes IP[k] when random bits are public. It turns out IP[k] $\subseteq$ AM[k + 2], but this is not obvious.

IP ⊆ PSPACE



- Consider k rounds of interaction:
  - $q_1 = V(x, r_1)$
  - . . .
  - $a_k = P(x, q_1, a_1 , . . . , q_{k-1}, a_{k-1}, q_k)$
  - $y = V(x, q_1, a_1 , . . . , q_{k-1}, a_{k-1}, q_k, r_k)$
- P is a PTIME function, in the absence of random bits, the probability of its output can be computed in PSPACE (just try every random sequence)
- Given $(x, q_1, a_1 , . . . , q_k)$, a_k is PSPACE computable:
  - Try each a_k, pick the one for which PROB[y = 1] is highest.
- Induction: since a_k is PSPACE computable, so is a_{k-1}:
  - Again, try each a_{k-1}, find a_k, maximize PROB[y = 1].

PSPACE Completeness



- It turns out IP = PSPACE! (Shamir, 1990)
- To show this, show a PSPACE complete language is in IP.
- TQBF: x is in TQBF if it represents a true ``Totally Quantified Boolean Formula'':
  - A boolean formula, $F(x_1, \ldots x_n)$, is a boolean expression comprised of AND's, ORs and NOTs.
  - In TQBF, each variable appears in either an existential or universal quantifier to the left of $F(x_1, \ldots x_n)$.
  - TQBF becomes SAT if quantifiers must be existential,
- TQBF is PSPACE Complete.

TQBF is PSPACE-Complete

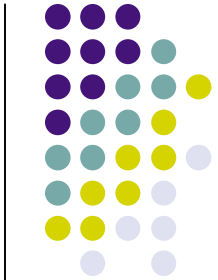
- To show SAT is NP complete, the computation (or tableau) of a nondeterminstic turning is represented as an existentially quantified boolean formula.
- To show TQBF is PSPACE complete, we represent a PSPACE computation using a quantified formula.
- Given a machine M, let $F_M(x,y,t)$ denote the boolean function ``M goes from state x to state y in t steps''.
- Notice that $F_M(x,y,t) = \exists w\ F_M(x,w,t/2) \wedge F_M(w,y,t/2)$.

  $= \exists w \forall x',y' \ ((x',y' = x,w) \wedge (x',y' = w,y)) \rightarrow F_M(x',y',t/2)$
- The formula $F_M(x,y,1)$ can be implemented directly when a polynomial number of variables represent x and y.
- If M uses a polynomial amount of space, $F_M(x,y,t)$ can be reduced to a polynomial sized instance of TQBF in PTIME.

Formulas As Polynomials

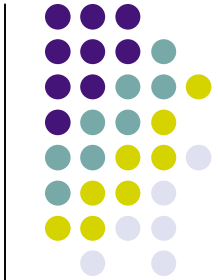
- A boolean expression can be represented as an arithmetic expression as follows:
  - $x \wedge y = (xy)$
  - $x \vee y = (x + y)$
  - $\neg x = (1 - x)$
- If variables are binary, we can also let
  - $\exists x\, f(x, y)$ denote $\sum_x f(x, y) - \prod_x f(x, y) = 1$
  - $\forall x\, f(x, y)$ denote $\prod_x f(x, y) = 1$
- Also, if variables are binary, $x^k = x$ for $k \geq 1$
  - The polynomial representing any boolean formula, or partially quantified boolean formula is multilinear.

An IP for TQBF



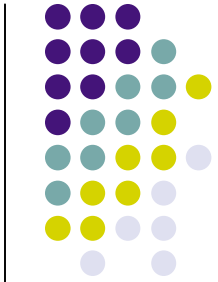
- We check if the arithmetic expression representing a TQBF with n variables evaluates to K = 0 or K = 1.
  - Arithmetic is mod some prime, p, provided by the prover.
- If n = 1
  - If universally quantified, check K' = f(0)f(1)
  - If existentially quantified, check K' = f(0) + f(1) - f(0)f(1)
- If n > 1
  - Let z be the variable quantified by the left most quantifier.
  - Ask the prover for a linear function, $h(z) = f(x_1,\ldots, z,\ldots, x_n)$
    - If z was universally quantified, check K' = h(0)h(1)
    - If existentially quantified, check K' = h(0) + h(1) - h(0)h(1)
  - Recurse: randomly r < p and check $f(r, x_2, \ldots, x_n) = h(r)$

Completeness and Soundess



- If the prover is truthful, it can respond to queries with $h(z) = (1 - z)\, f(x_1,\ldots, z,\ldots, x_n) + z\, f(x_1,\ldots, z,\ldots, x_n)$.
- When n = 1, the verifier rejects an incorrect value of K with probability 1.
- If the prover is not truthful for some n > 2
  - $h(z) \neq (1 - z)\, f(x_1,\ldots, z,\ldots, x_n) + z\, f(x_1,\ldots, z,\ldots, x_n)$.
  - Both functions are linear, so they agree on at most one value of z. The probability the verifier picks this as r is 1/p.
  - The verifier rejects an incorrect h(z) with probability at least $P_n \leq (1 - 1/p)P_{n-1} \leq (1 - 1/p)^{n-1}$.
- If p > 2n, P_n is sufficiently small.

Summarizing IP

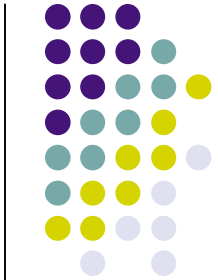


- TQBF is PSPACE-Complete
- TQBF is in IP with completeness 1
  - Recall that IP with arbitrary completeness is itself contained in PSPACE.
- It turns out IP remains the same even when random bits are public.
  - This is proven using hash functions. They allow a verifier to check if a set is at least size |S| or at most size |S|/2.
- The provers for TQBF and GI need only be able to solve instances of TQBF and GI, repsectively.
  - Using an interactive proof, a program solving these problems can be used to check itself.

MIP

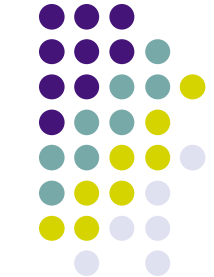
- The power of interactive proofs can be increased with multiple provers (Ben-Or, Goldwassert, Kilian, Wigdemons, 1988).
- For languages in MIP, the verifier can query two provers that cannot communicate.
  - The verifier can pit the provers against each other.
- Two provers can be used to recognized languages in IP in only a single round (Cai, Condon, Lipton, 1990)
- In fact, MIP = NEXPTIME (Babai, Fortnow, Lund, 1991)
  - Again, a single round of interaction suffices (Feige, 1991)
- Three or more provers can be simulated using 2 provers (Fortnow, Rompel, Sipser)

MIP and PCP



- MIP is incredibly powerful, but what if we keep questions short (O(log(n))?
- The PCP theorem implies NP-Complete problems can be reduced to gap instances of 3SAT.
- We have short a two prover one round protocol:
  - Ask P_1 the assignment to the three variables in a clauses
  - Ask P_2 the assignment to one of the variables
  - Check for disagreement.
- Multiple repetitions will increase soundness.
  - What about asking multiple questions at once? Yes!
  - This is called parallel repetition (Raz, 1998)

Conclusion



- In addition to PCP and NP, IP and MIP show that interaction is a powerful tool in classifying problems.
- PSPACE is equivalent to a polynomial number of PTIME interactions with a prover.
- NEXPTIME is equivalent to a constant number of PTIME interactions with three provers or a polynomial number of interactions with two.
- Next time we use PCPs to implement a two prover 1 round interactive proof. This leads to strong hardness of approximation results.