Homework 4

Due: 16 October, 2020

Each problem is graded on the coarse scale of $\checkmark^+$, $\checkmark$, $\checkmark^-$ and no $\checkmark$. It is also assigned a multiplier, denoting the relative importance of the problem. Both correctness and presentation are grading criteria.

Please read and make sure you understand the collaboration policy on the course missive. The problems for 2000-level credit are clearly marked below: students who are registered for 1000-level credit are welcome to solve and submit these problems for extra credit (see course missive for details).

Remember to prove all your (non-elementary) mathematical claims, unless stated otherwise.

Each pair of students should submit only 1 pdf to the corresponding Canvas assignment.

Problem 1

(5 $\checkmark$s total)

Recall the notation that $\mathsf{PCP}[r(n), q(n)]$ denotes the class of languages with probabilistically checkable proof systems that have non-adaptive verifiers using $r(n)$ bits of randomness and $q(n)$ bits of queries, with completeness probability 1 and soundness probability $\frac{1}{2}$.

For each of the following, state and prove what complexity class it denotes (you don't need to have taken a complexity theory course for this, you have seen all these classes already).

Wikipedia has the answer to all this, but try solving this on your own! Also remember that you have to prove your answers, which Wikipedia doesn't do for you.

1. $\mathsf{PCP}[0, 0]$
2. $\mathsf{PCP}[O(\log n), 0]$
3. $\mathsf{PCP}[0, O(\log n)]$
4. $\mathsf{PCP}[O(\log n), O(1)]$
5. $\mathsf{PCP}[O(\log n), O(\log n)]$
6. $\mathsf{PCP}[0, \mathrm{poly}(n)]$
7. $\mathsf{PCP}[O(\log n), \mathrm{poly}(n)]$

Problem 2

The aim of this problem is to use the Szemerédi regularity lemma to show that the number of triangle-free graphs over n vertices is $2^{(\frac{1}{4}+o(1))n^2}$. Note that, for this problem, graphs that are isomorphic to each other is still counted as distinct graphs.

(a) (1 ✓) Show that there are at least $2^{\frac{n^2}{4}}$ triangle-free graphs over n vertices.

(b) (2 ✓s) Show, by induction or otherwise, that all triangle-free graphs have at most $\frac{n^2}{4}$ edges.

(c) (3 ✓s, **2000-level**) Using part (b) and the regularity lemma, show that there can be at most $2^{(\frac{1}{4}+o(1))n^2}$ triangle-free graphs. (Hint: $2^{o(n^2)}$ is a very large multiplicative factor that is slack in this result, so be generous in overcounting.) You may use the following fact without proof:

$$\binom{k}{\alpha k} \leq 2^{k(H(\alpha)+o(1))} \text{ where } H(\alpha) \text{ is a function that tends to 0 as } \alpha \text{ tends to 0}$$

For context, if you know any information theory (or large deviation bounds), the function $H(\alpha)$ is simply the entropy of a coin with bias α.