Differential Equations for Modeling Hybrid Systems

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1 A note on notation

For a function $f(t)$, you may have seen the following notations to denote the function's first derivative:

$$\dot{f}(t) \equiv f'(t) \equiv \frac{df}{dx}.$$

The second derivative can similarly be written

$$\ddot{f}(t) \equiv f''(t) \equiv \frac{d^2 f}{dx^2}.$$

In this document, as in Lee/Seshia, we use dot notation ($\dot{f}(t)$) to denote the derivative.

2 Derivatives

In physics and calculus, we use derivatives to describe the rate of change of a function. For example, if the displacement d over time t of an object moving in one dimension is $d(t) = 5 + 2t$, its velocity v will be the derivative

$$v(t) = \dot{d}(t) = 2.$$

Its acceleration will be the derivative of the velocity (or the second derivative of the position):

$$a(t) = \dot{v}(t) = \ddot{d}(t) = 0.$$

3 Differential equations and initial conditions

In some cases, it is desirable to describe a function implicitly by some initial condition $f(0) = \alpha$ and its rate of change $\dot{f}(t) = w(t)$, where w is some function on t. The equation $\dot{f}(t) = w(t)$ is called an *Ordinary Differential Equation (ODE).*

For example, say we have a function y with initial condition $y(t) = 0$ and ODE $\dot{y}(t) = 1$. Then, the following explicit form of y satisfies both the initial condition and the ODE:

$$y(t) = t,$$

which you can verify by substituting t for 0 to check the initial condition and differentiating y to check the ODE.

In some cases, you can solve for the explicit form of a function by integrating. For example, if $\dot{x}(t) = 6t + 4$,

$$x(t) = \int \dot{x}(t) = \int 6t + 4 \; dt = 3t^2 + 4t + C,$$

where C is a constant. If we are given the initial condition $x(0) = -1$, then we can solve for C:

$$x(0) = 3 \cdot 0^2 + 4 \cdot 0 + C = -1 \implies C = -1.$$

Differential equations can also apply to higher derivatives. For example, given the conditions

$$w(0) = 0, \dot{w}(0) = 0, \ddot{w}(t) = 2,$$

we can solve for an explicit form of w:

$$w(t) = t^2.$$

(Verify for yourself that this is true.)

3.1 ODEs that depend on the function

Some ODEs are written in terms of the original function. For example,

$$\dot{g}(t) = \beta g(t)$$

is satisfied when $g(t) = \gamma e^{\beta t}$, for any constant γ. If we are given the initial condition $g(0) = 1$, then we know $\gamma = 1$.

Sometimes, ODEs may have multiple solutions of more than one form:

$$\ddot{h}(t) = -h(t)$$

is satisfied by both $h(t) = \xi sin(t)$ and $h(t) = \xi cos(t)$ for any constant ξ. An initial condition can sometimes tell us which form applies: if $h(0) = 1$, then we know $h(t)$ must be $cos(t)$, because $sin(0) = 0$. However, if we are given a different initial condition $h(\pi/6) = 2$, then we do not know which form applies, because both $\frac{4}{\sqrt{3}}cos(t)$ and $4sin(t)$ satisfy that initial condition.

Learning how to solve different kinds of ODEs takes up a course's (or more) worth of material and is beyond the scope of this document. However, after reading this document, you should have the background necessary to reason about the sorts of hybrid system modeling we will do in class.