Partner 1
Partner 2 CSCI 1550 / 2540
Partner 3 March 20th, 2025

# Homework 4

*Due:* April 10th, 2025
Remember to show your work for each problem to receive full credit.

## Problem 1 (30 points)

1. Let $X_1, X_2, \ldots,$ be a sequence of independent exponential random variables, each with mean 1. Given a positive real number $k$, let $N$ be defined by

$$N = \min \left\{ n : \sum_{i=1}^{n} X_i > k \right\}.$$

That is, $N$ is the smallest number for which the sum of the first $N$ of the $X_i$ is larger than $k$. Use Wald's equation to determine $E[N]$.

2. Let $X_1, X_2, \ldots,$ be a sequence of independent uniform random variables on the interval $(0, 1)$. Given a positive real number $k$, with $0 < k < 1$, let $N$ be defined by

$$N = \min \left\{ n : \prod_{i=1}^{n} X_i < k \right\}.$$

That is, $N$ is the smallest number for which the product of the first $N$ of the $X_i$ is smaller than $k$. Determine $E[N]$. (Hint: Prove that $\log 1/X_i$ has an exponential distribution.)

# Problem 2 (40 points)

A random graph $G_{n,m}$ has $n$ vertices and $m$ edges. The $m$ edges are chosen uniformly at random (without repetition) among all the possible $\binom{n}{2}$ edges. Consider a random graph $G_{n,m}$, where $m = cn$ for some constant $c > 0$. Let $X$ be the number of isolated vertices (i.e., vertices of degree 0).

1. Compute $E(X)$
2. For any $\lambda > 0$, show that $\Pr(|X - E(X)| \geq 2\lambda\sqrt{cn}) \leq 2e^{-\lambda^2/2}$ (*Hint*: Use a martingale that reveals the locations of the edges in the graph, one at a time).

# Problem 3 (25 points)

Consider a random walk on the infinite two dimension integer grid:

$$G = \{(x, y) \mid x \in \{-\infty, \infty\},\ y \in \{-\infty, \infty\}\}.$$

The random walk starts at $(0, 0)$, and if the walk is at $(x_t, y_t)$ at time $t$, then with equal probabilities the walk moves to one of the adjacent nodes $(x_t - 1, y_t)$, $(x_t, y_t - 1)$, $(x_t + 1, y)$, or $(x_t, y_t + 1)$. I.e.

$$Pr((x_{t+1}, y_{t+1})) \mid (x_t, y_t)) = \begin{cases} 1/4 & \text{if} \quad (x_{t+1}, y_{t+1}) = (x_t - 1, y_t) \\ 1/4 & \text{if} \quad (x_{t+1}, y_{t+1}) = (x_t + 1, y_t) \\ 1/4 & \text{if} \quad (x_{t+1}, y_{t+1}) = (x_t, y_t - 1) \\ 1/4 & \text{if} \quad (x_{t+1}, y_{t+1}) = (x_t, y_t + 1) \end{cases}$$

Prove that for $\lambda > 0$

$$Pr\left(|x_t + y_t| \geq \lambda\sqrt{t}\right) \leq 2e^{-\lambda^2/2}.$$

## Problem 4 (25 points)

Let $f(X_1, X_2, \ldots, X_n)$ satisfy the Lipschitz condition so that, for any $i$ and any values $x_1, \ldots, x_n$ and $y_i$,

$$|f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)| \leq c.$$

We set

$$Z_0 = \mathbb{E}(f(X_1, \ldots, X_n))$$

and

$$Z_i = \mathbb{E}(f(X_1, \ldots, X_n) \mid X_1, \ldots, X_i).$$

Give an example to show that, if the $X_i$ are not independent, then it is possible that $|Z_i - Z_{i-1}| > c$ for some $i$.

# Problem 5 (30 points)

Consider a bin with $N > 1$ balls. The balls are either black or white. Let $X_0 = \frac{m}{N} < 1$ be the fraction of black balls in the bin at time 0. Let $X_i$ be the fraction of black balls at time $i$. At step $i \geq 1$ one ball, chosen uniformly at random, is replace with a new ball. With probability $X_i$ the new ball is black, otherwise it is white. All random choices are independent.

Consider the stopping time $\tau := \inf_i \{X_i \in \{0, 1\}\}$. That is, the process stops when all balls have the same color.

(a) Show that $X_1, X_2, \ldots$ is a martingale with respect to itself.

(b) Show that $E[\tau] < \infty$.
**Hint:** Show that at any step there is probability $\geq (\frac{1}{2N})^{N/2}$ to terminate in the next $N/2$ steps. Conclude that $E[\tau] \leq (2N)^{N/2} \cdot N/2$.

(c) Calculate $\mathbb{P}(X_\tau = 1)$