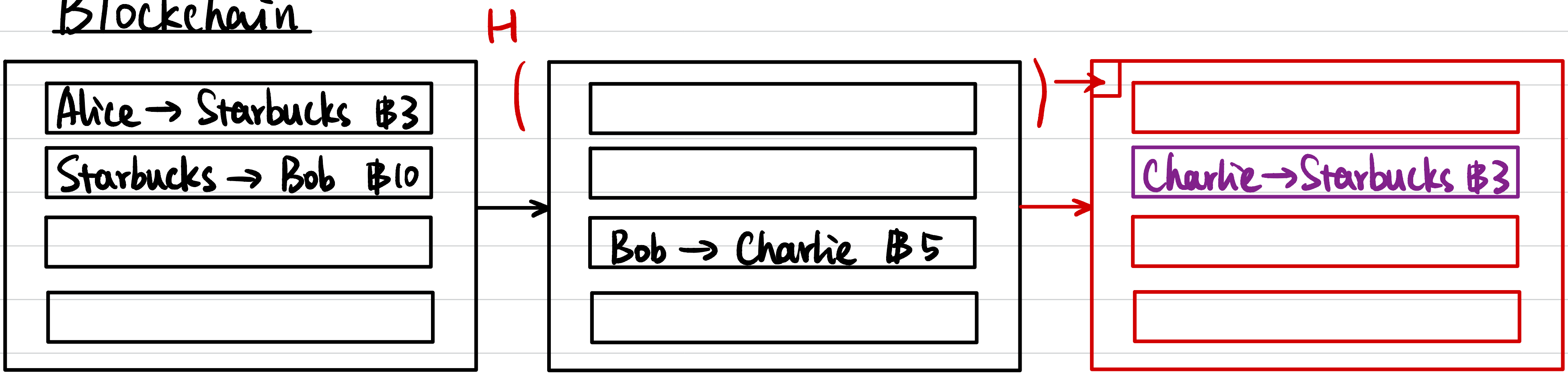
CSCI 1515 Applied Cryptography

This Lecture:

- Blockchain & Cryptocurrencies (Continued)
- Elliptic Curve Cryptography

Blockchain



- **Public** ledger that everyone can view & verify
- Maintained by "miners" in a **distributed** way

Step 1: Charlie wants to make a transaction Charlie → Starbucks ฿3

↳ broadcasts it to the entire network

Step 2: All miners collect all transactions in the network

- Verify validity
  1. initiated by sender ← Digital Signatures
  2. enough balance in sender's account
- Agree on next block ← How?

Step 3: Repeat

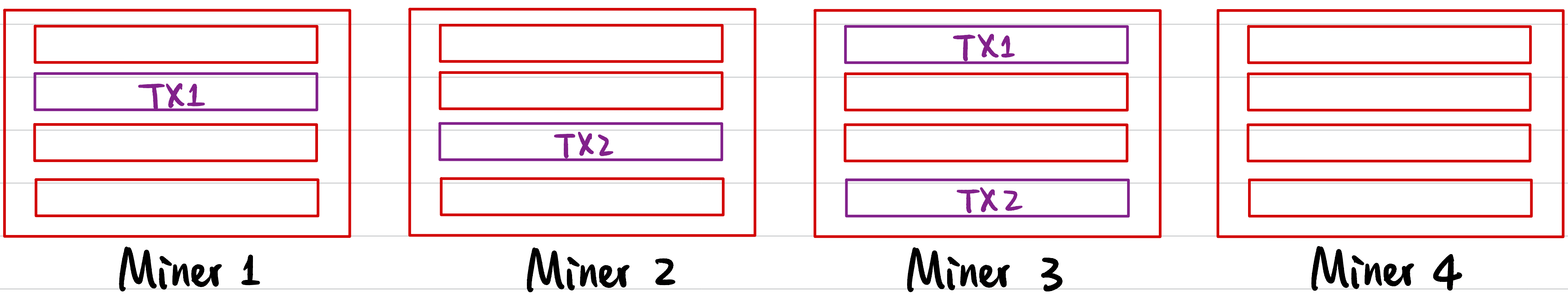
Consensus Protocol

TX1 = $\boxed{\text{Charlie} \to \text{Starbucks ฿3}}$:

$$m_2 = (vk_C,\ vk_S,\ 3) \qquad \sigma_2 \leftarrow Sign_{sk_C}(m_2)$$

TX2 = $\boxed{\text{Charlie} \to \text{Alice ฿4}}$:

$$m_3 = (vk_C,\ vk_A,\ 4) \qquad \sigma_3 \leftarrow Sign_{sk_C}(m_3)$$

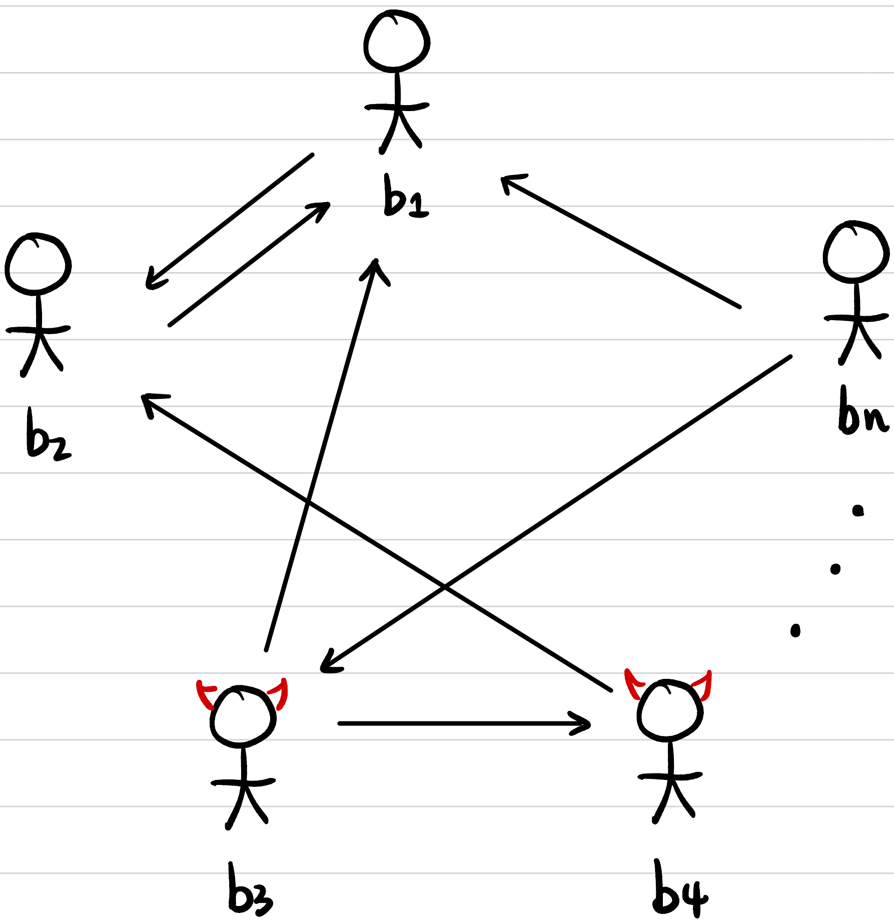


"permissionless"

WANT: ① All miners agree on the same block

② New block is valid

Byzantine Agreement



⇓

Agree on a block

Byzantine Fault Tolerance (BFT) Protocol:

necessary

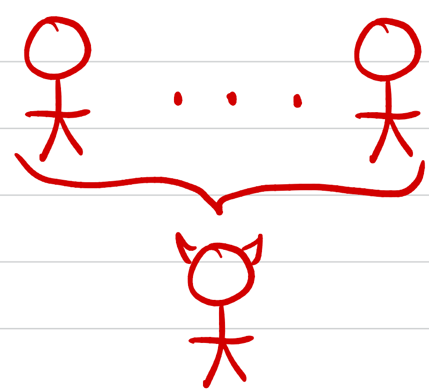
If $n \geq 3t+1$,

then it's possible to reach consensus.

Assume $t < n/3$, then agree on a valid block.

Any problem?

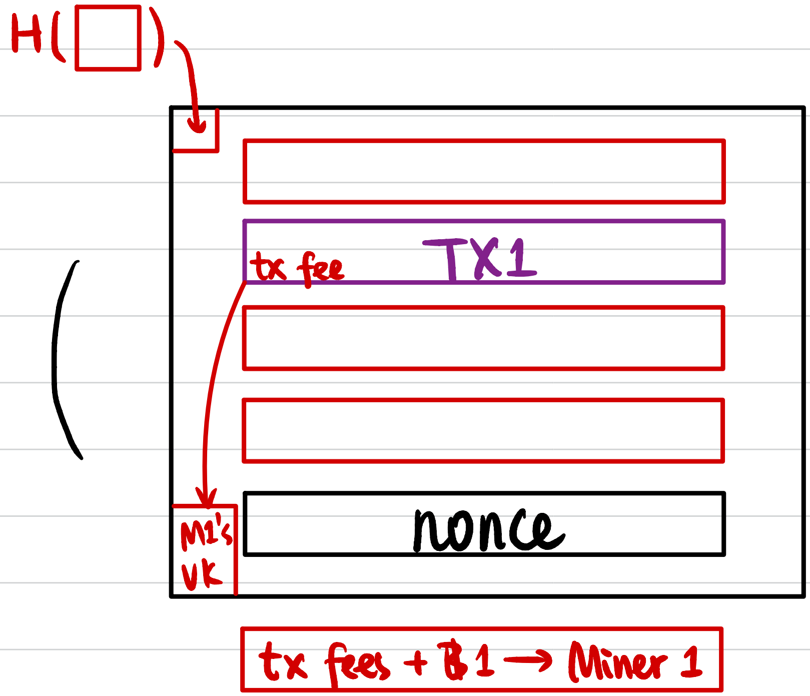
"Sybil Attack"



Proof of Work (PoW)

Miner 1:

Hash (



)

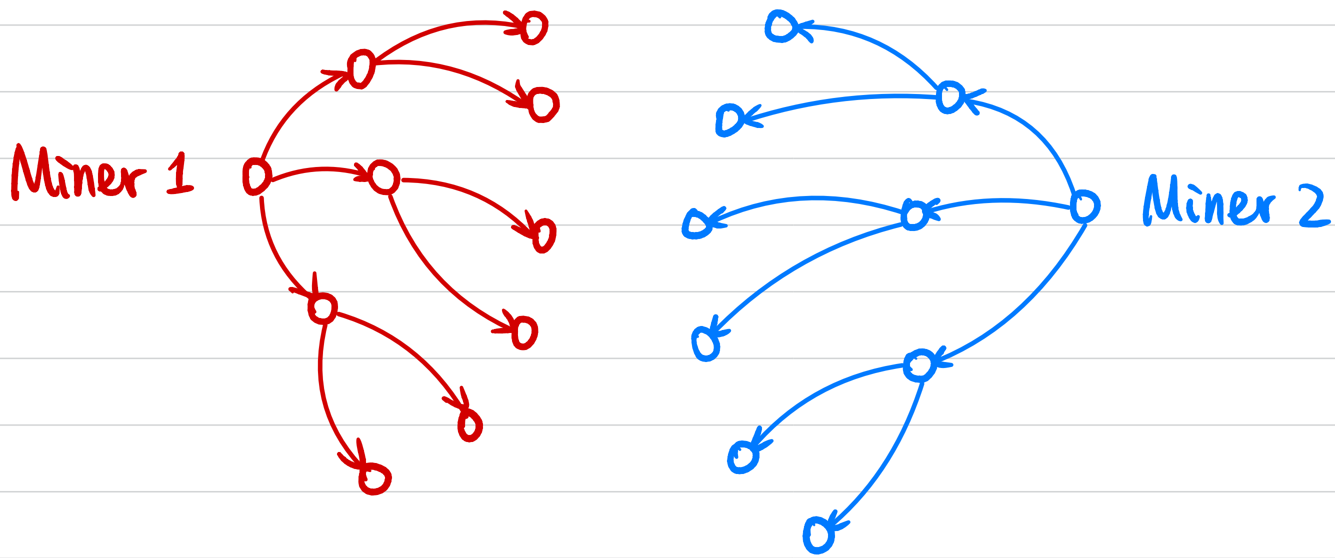
$$= \underbrace{00\cdots0}_{30}1011\cdots0$$

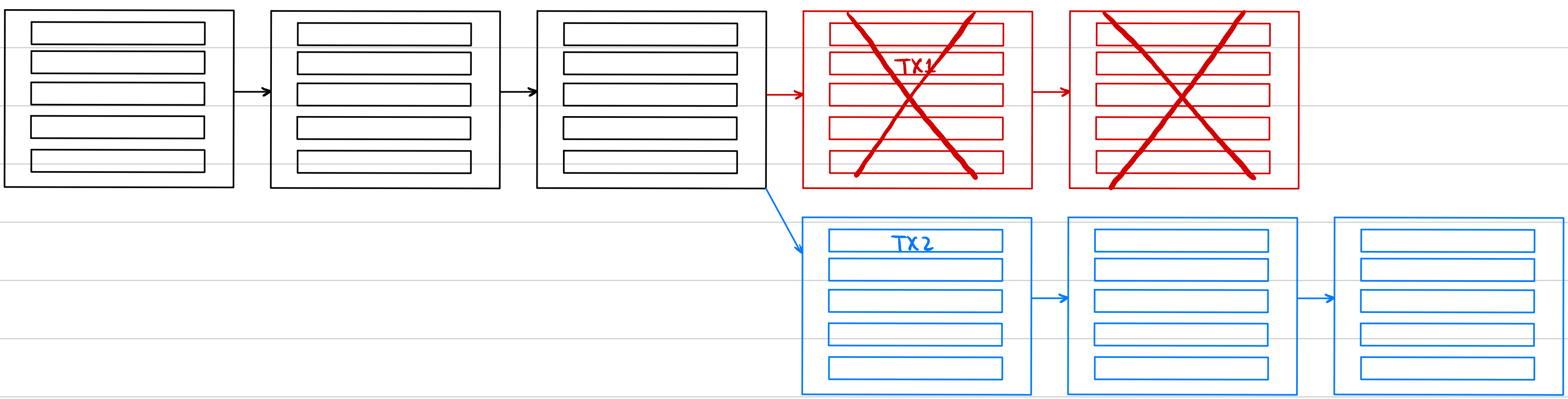
Find nonce s.t. Hash(block) has ≥ 30 leading 0's.

Consensus Protocol:

Whoever first finds a block that hashes to a value w/ ≥ 30 leading 0's, that block becomes the next block.

Proof of Work (PoW)





Longest Chain Rule: Always adopt the longest chain.

Assuming honest majority of computation power, the longest chain is always valid.

Blockchain

- Efficient verification of sufficient balance: Merkle Tree

- Settlement of a transaction:
  Included in a block which is ≥ 6 blocks deep (~ 1 hr)

- Dynamically adjust # leading 0's s.t. each block takes ~10min to mine
  Last 1 hr: > 6 blocks: increase # leading 0's
  < 6 blocks: decrease # leading 0's

- Miners' motivation:
  - transaction fee
  - new coin generated in each block goes to miner

- Extensions
  - Fast verification (SNARGs)
  - Proof of Stake (PoS)
  - Anonymous transactions (zk-SNARGs)
  - Smart Contracts
  - Public Bulletin Board

Elliptic Curve Cryptography

Cyclic group $\mathbb{G}$ of order q with generator g where DLOG / CDH / DDH holds.

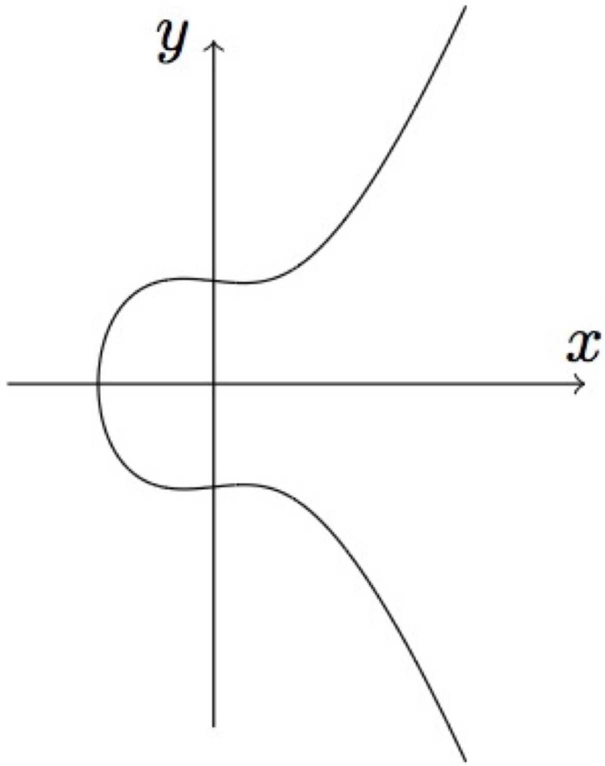
$\uparrow$

How large is q ? (128-bit security)

- Integer groups: $q \sim 2048$ bits
- Elliptic curve groups: $q \sim 256$ bits

$\hookrightarrow$ Additional structure: bilinear pairings

Elliptic Curves



$$y^2 = x^3 + ax + b$$

$$(4a^3 + 27b^2 \neq 0)$$

Example: $y^2 = x^3 - x + 9$

points: $(0, \pm 3)$
$(1, \pm 3)$
$(-1, \pm 3)$

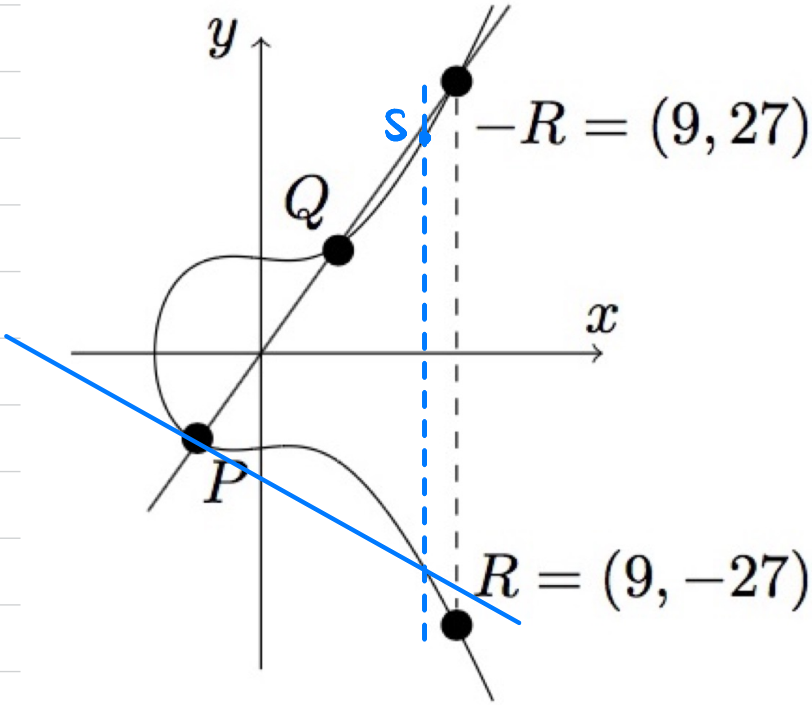
How to find rational points $(x, y) \in \mathbb{Q}^2$ on the curve?

$$x = \frac{s}{t}, \quad y = \frac{u}{v}$$

$$s, t, u, v \in \mathbb{Z}$$

Elliptic Curves

How to find rational points $(x, y) \in \mathbb{Q}^2$ on the curve?



Example: $y^2 = x^3 - x + 9$

① Chord method $R := P \boxplus Q$

$$P = (-1, -3), \quad Q = (1, 3) \Rightarrow y = 3x$$

$$\Downarrow$$

$$(3x)^2 = x^3 - x + 9$$

$$x^3 - 9x^2 - x + 9 = 0$$

Why is the third root rational?

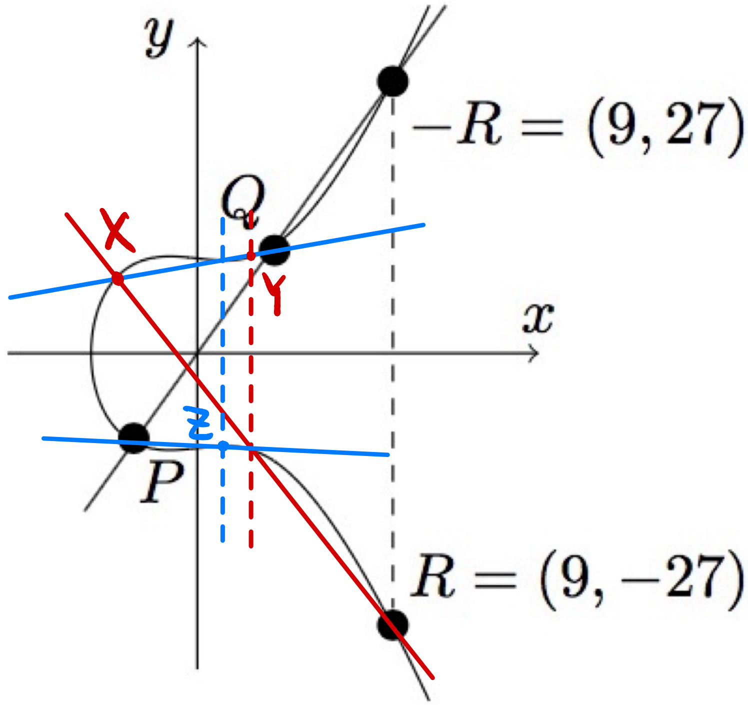
$$(x - x_1)(x - x_2)(x - x_3) = 0$$

$$(+1)(-1)(-x_3) = 9$$

$$x_3 = \frac{-9}{(+1)\cdot(-1)} = 9$$

② tangent method $S := P \boxplus P$

Elliptic Curves



$R := P \boxplus Q$

$(P \boxplus Q) \boxplus X = P \boxplus (Q \boxplus X)$

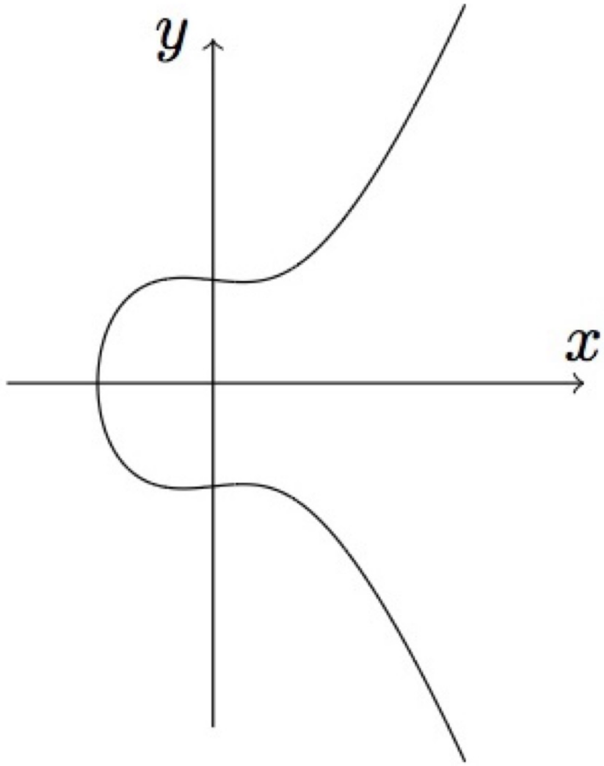
$R = P \boxplus Q$ $\quad$ $Z = Q \boxplus X$

$Y = R \boxplus X$ $\quad$ $Y = P \boxplus Z$

$P \boxplus Q = Q \boxplus P$

Example: $y^2 = x^3 - x + 9$

Elliptic Curves over Finite Fields



$$y^2 = x^3 + ax + b$$

$$(4a^3 + 27b^2 \neq 0)$$

$\{0, 1, \cdots, p-1\}$, $+$, $\cdot$, inverse

Finite field $\mathbb{F}_p$, $p > 3$ prime

Elliptic cure E defined over $\mathbb{F}_p$: $E/\mathbb{F}_p$.

$a, b \in \mathbb{F}_p$

(x, y) is a point on the cure if

$x, y \in \mathbb{F}_p$

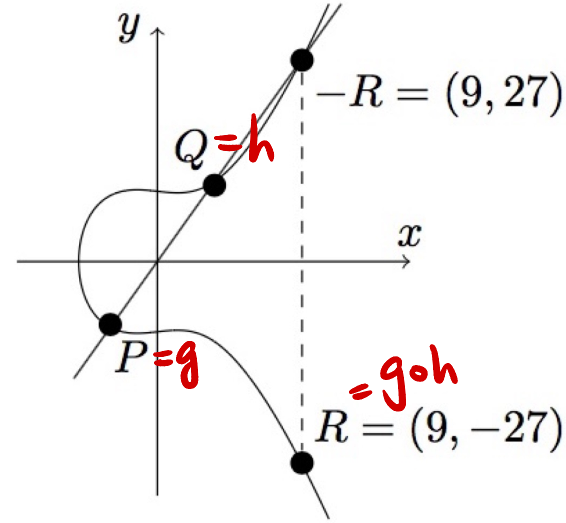
$y^2 = x^3 + ax + b$ over $\mathbb{F}_p$

Point at infinity: O

Example: $y^2 = x^3 + 1$ over $\mathbb{F}_{11}$.

$$E/\mathbb{F}_{11} = \{O, (-1, 0), (0, \pm 1), (2, \pm 3), (5, \pm 4), (7, \pm 5), (9, \pm 2)\}$$

Elliptic Curves over Finite Fields



Group properties:

① **Closure:** $\forall g, h \in G,\ \underset{\boxplus}{g \circ h} \in G$

② **Existence of an identity:** Point at infinity: O

$\exists e \in G$ s.t. $\forall g \in G,\ e \circ g = g \circ e = g.$ $\quad g \boxplus O := g$

③ **Existence of inverse:**

$\forall g \in G,\ \exists h \in G$ s.t. $g \circ h = h \circ g = e$ $\quad h := -g$

④ **Associativity:**

$\forall g_1, g_2, g_3 \in G,\ (g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$

⑤ **Commutativity (abelian):**

$\forall g, h \in G,\ g \circ h = h \circ g$

SEA algorithm: Count number of points on $E/\mathbb{F}_p$ in time $\mathrm{polylog}(p)$.

How to compute g^a for $a \in \mathbb{Z}_q$? $\quad \underbrace{g \boxplus g \boxplus \cdots \boxplus g}_{a}$

Elliptic Curve Cryptography

- curve secp256r1 (P256)
  - prime $p = 2^{256} - 2^{224} + 2^{192} + 2^{96} - 1$
  - $y^2 = x^3 - 3x + b$ b: 255-bit
  - Number of points on the curve is prime (close to p)
  - Generator point G

- Curve secp256k1

- Curve25519