CSCI 1515 Applied Cryptography

This Lecture:

- RSA Assumption/Encryption (Continued)
- Diffie-Hellman Assumptions
- ElGamal Encryption
- Diffie-Hellman Key Exchange
- Message Integrity
- RSA Signature

Basic Number Theory

$\gcd(a, N) = 1$: a & N are coprime

$\Rightarrow \exists b$ s.t. $a \cdot b \equiv 1 \pmod{N}$: a is invertible modulo N,
b is its inverse, denoted as a^{-1}.

$\mathbb{Z}_N^* := \{a \mid a \in [1, N-1], \gcd(a, N) = 1\}$

Euler's phi (totient) function $\phi(N) := |\mathbb{Z}_N^*|$

Ex: $N = p \cdot q$ (p, q are primes) $\phi(N) = (p-1) \cdot (q-1)$.

Euler's Theorem $\forall a, N$ where $\gcd(a, N) = 1$, $a^{\phi(N)} \equiv 1 \pmod{N}$.

Corollary If $d \equiv e^{-1} \bmod \phi(N)$, then $\forall a \in \mathbb{Z}_N^*$, $(a^d)^e \equiv a \bmod N$.

RSA Assumption

- Factoring Assumption:

  How?

  Generate two n-bit primes p, q $(p \neq q)$

  Compute $N = p \cdot q$

  Given N, it's computationally hard to find p & q (classically).

- RSA Assumption:

  Generate two n-bit primes p, q $(p \neq q)$

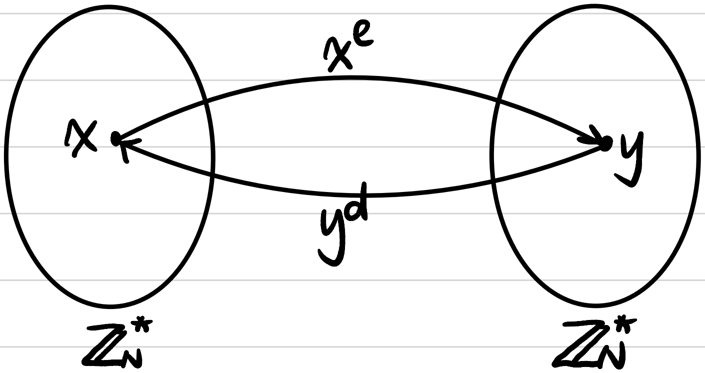
  Compute $N = p \cdot q$, $\phi(N) = (p-1)(q-1)$

  Choose e s.t. $\gcd(e, \phi(N)) = 1$

  Compute $d = e^{-1} \mod \phi(N)$.

  Given N & a random $y \xleftarrow{\$} \mathbb{Z}_N^*$, it's computationally hard to find x s.t.

  $$x^e \equiv y \pmod{N}$$



"Plain" RSA Encryption

$\lambda = 128$ (best attack takes time 2^{128})

$n = 1024$, key length 2048

- $\mathrm{Gen}(1^\lambda)$:

  $n = O(\lambda)$

  Generate two n-bit p, q $(p \neq q)$

  Compute $N = p \cdot q$, $\phi(N) = (p-1)(q-1)$

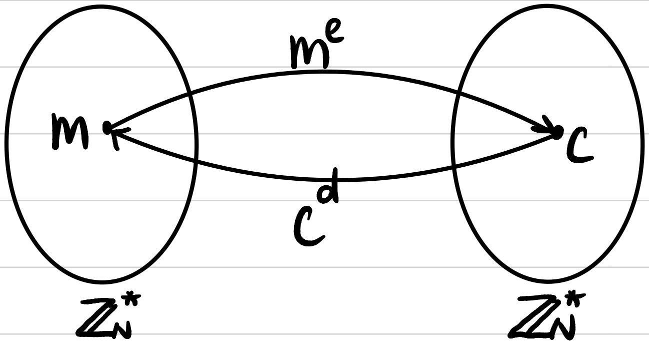
  Choose e s.t. $\gcd(e, \phi(N)) = 1$

  Compute $d = e^{-1} \bmod \phi(N)$.

  $pk = (N, e)$ $\quad$ $sk = d$.

- $\mathrm{Enc}_{pk}(m)$: $c = m^e \bmod N$

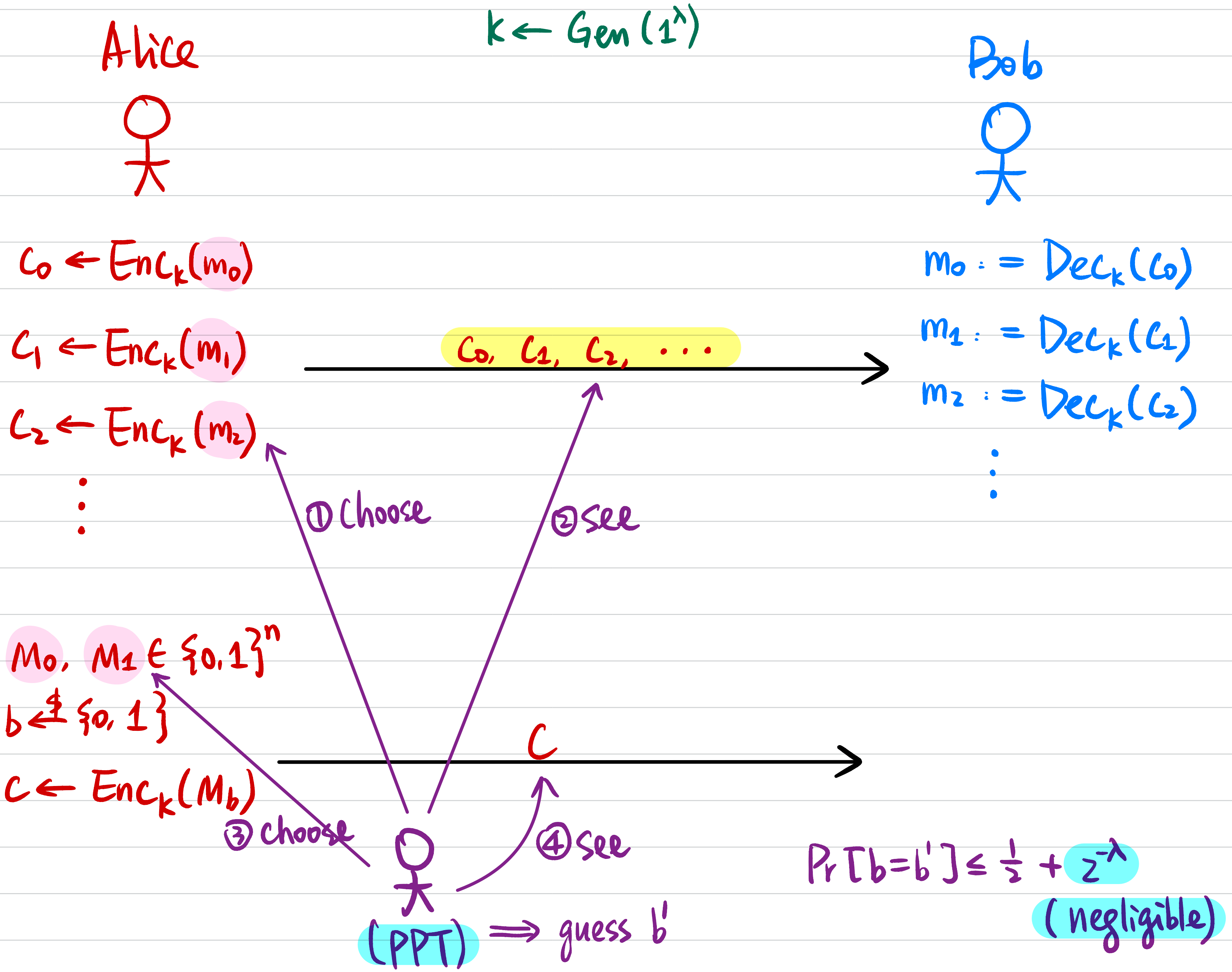
- $\mathrm{Enc}_{sk}(c)$: $m = c^d \bmod N$



Any security issue?

Computational Security

Chosen-Plaintext Attack (CPA) Security



$k \leftarrow Gen(1^\lambda)$

$c_0 \leftarrow Enc_k(m_0)$

$c_1 \leftarrow Enc_k(m_1)$

$c_2 \leftarrow Enc_k(m_2)$

$\vdots$

$m_0 := Dec_k(c_0)$

$m_1 := Dec_k(c_1)$

$m_2 := Dec_k(c_2)$

$\vdots$

$M_0, M_1 \in \{0,1\}^n$

$b \overset{\$}{\leftarrow} \{0,1\}$

$c \leftarrow Enc_k(M_b)$

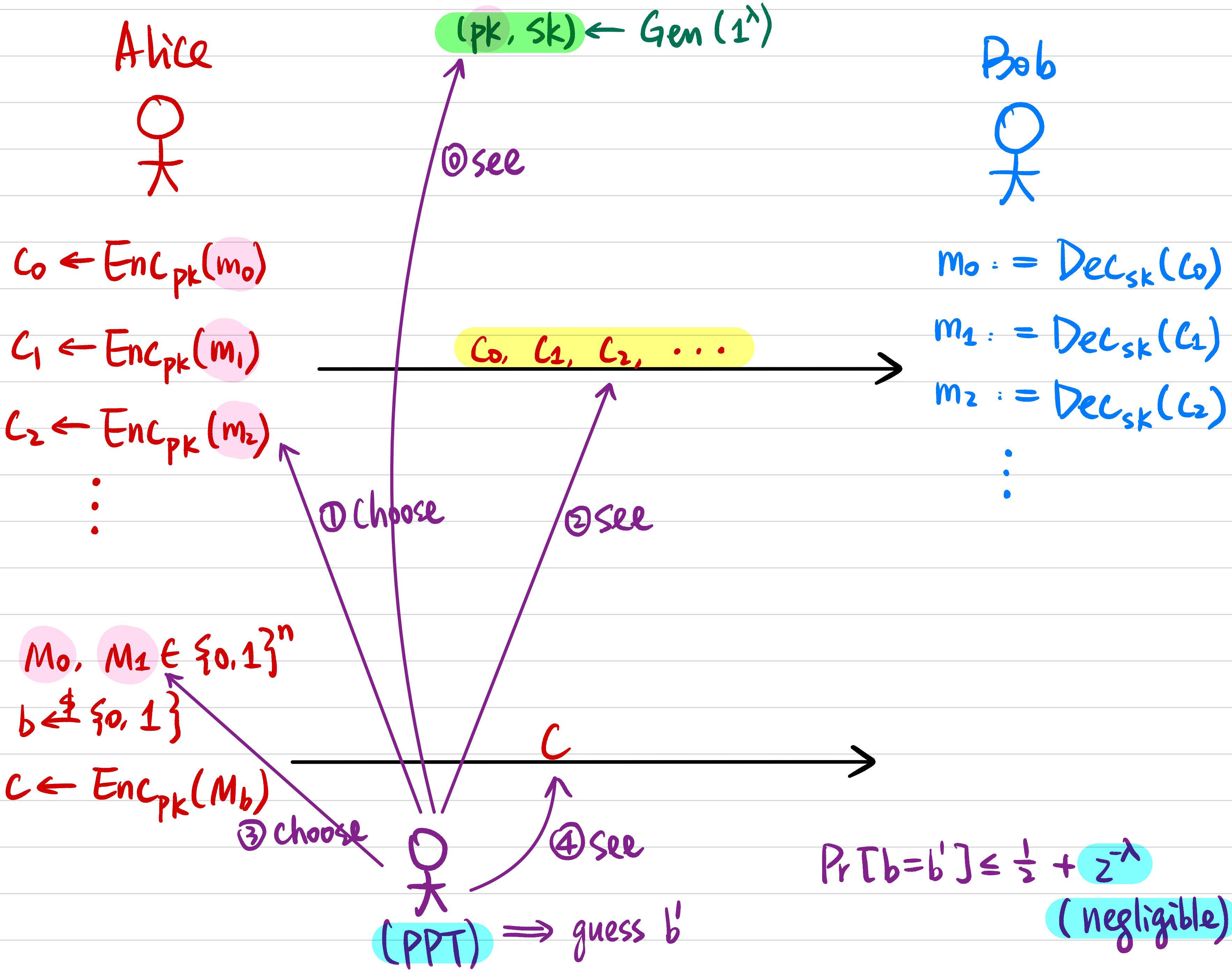
$\Longrightarrow$ guess b'

$Pr[b=b'] \leq \frac{1}{2} + 2^{-\lambda}$

(negligible)

Computational Security

Chosen-Plaintext Attack (CPA) Security



$Pr[b=b'] \leq \frac{1}{2} + 2^{-\lambda}$

(negligible)

Basic Group Theory

Def A group is a set G along with a binary operation $\circ$ with properties:

① Closure: $\forall g, h \in G,\ g \circ h \in G$

② Existence of an identity: $\exists e \in G$ s.t. $\forall g \in G,\ e \circ g = g \circ e = g$.

③ Existence of inverse: $\forall g \in G,\ \exists h \in G$ s.t. $g \circ h = h \circ g = e$

Inverse of g denoted as g^{-1}.

④ Associativity: $\forall g_1, g_2, g_3 \in G,\ (g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$

We say a group is abelian if it satisfies:

⑤ Commutativity: $\forall g, h \in G,\ g \circ h = h \circ g$

Exercises: Is this a group?

- $(\mathbb{Z}, +)$
- $(\mathbb{Z}, \cdot)$
- $(G = \{0, 1, \cdots, N-1\},\ +\ \mathrm{mod}\ N)$
- $(\mathbb{Z}_N^*,\ \cdot\ \mathrm{mod}\ N)$

Basic Group Theory

Group Exponentiation

For a group $(G, \cdot)$, $g^m := \overbrace{g \cdot g \cdots g}^{m}$ $\quad g^0 := 1$ $\quad g^{-m} := (g^{-1})^m$

$g^{m_1} \cdot g^{m_2} = g^{m_1+m_2}$ $\quad (g^{m_1})^{m_2} = g^{m_1 \cdot m_2}$ $\quad g^m \cdot h^m = (g \cdot h)^m$ $\quad g^{-m} = (g^m)^{-1}$

Def For a finite group, we use $|G|$ to denote its order (# of elements)

Let G be a finite group of order m.

$\forall g \in G$, $\langle g \rangle := \{g^0, g^1, \cdots, g^{m-1}\}$ $\quad (g^m = 1)$ $\quad \underbrace{1, g, g^2, \cdots}_{|\langle g \rangle|}, \underbrace{1, g, g^2, \cdots}, \cdots \; g^m = 1$

G is a cyclic group if $\exists g \in G$ s.t. $\langle g \rangle = G$. g is a generator of G.

Thm $\mathbb{Z}_p^*$ for a prime p is a cyclic group of order $p-1$.

$p = 7$, $\langle 3 \rangle = \{1, 3, 2, 6, 4, 5\}$ $\quad \langle 2 \rangle = \{1, 2, 4\}$

Thm If G has prime order, then G is cyclic and every element except the identity is a generator.

Diffie-Hellman Assumptions

Integer group key 2048-bit
Elliptic Curve group key 256-bit

$(\mathbb{G}, q, g) \leftarrow \mathcal{G}(1^\lambda)$

$O(\lambda)$-bit integer

Cyclic group $\mathbb{G}$ of order q with generator g

- **Discrete Logarithm (DLOG)** Assumption:

  $x \xleftarrow{\$} \mathbb{Z}_q$, compute $h = g^x$ $\quad g^x \overset{?}{\Longrightarrow} x$

  Given $(\mathbb{G}, q, g, h)$, it's computationally hard to find x (classically).

- **Computational Diffie-Hellman (CDH)** Assumption:

  $x, y \xleftarrow{\$} \mathbb{Z}_q$, compute $h_1 = g^x$, $h_2 = g^y$ $\quad (g^x, g^y) \overset{?}{\Longrightarrow} g^{xy}$

  Given $(\mathbb{G}, q, g, h_1, h_2)$, it's computationally hard to find g^{xy}.

- **Dicisional Diffie-Hellman (DDH)** Assumption:

  $x, y, z \xleftarrow{\$} \mathbb{Z}_q$, compute $h_1 = g^x$, $h_2 = g^y$

  Given $(\mathbb{G}, q, g, h_1, h_2)$, it's computationally hard to distinguish between g^{xy} and g^z.

  $$(g^x, g^y, g^{xy}) \overset{c}{\approx} (g^x, g^y, g^z)$$

ElGamal Encryption

- $\mathrm{Gen}(1^\lambda)$:

  $(\mathbb{G}, q, g) \leftarrow \mathcal{G}(1^\lambda)$ ← can be re-used

  $x \xleftarrow{\$} \mathbb{Z}_q$, compute $h = g^x$

  $pk = (\mathbb{G}, q, g, h) \quad sk = x$

- $\mathrm{Enc}_{pk}(m)$: $m \in \mathbb{G}$

  $y \xleftarrow{\$} \mathbb{Z}_q$

  $c = \langle g^y, h^y \cdot m \rangle$

- $\mathrm{Enc}_{sk}(c)$:

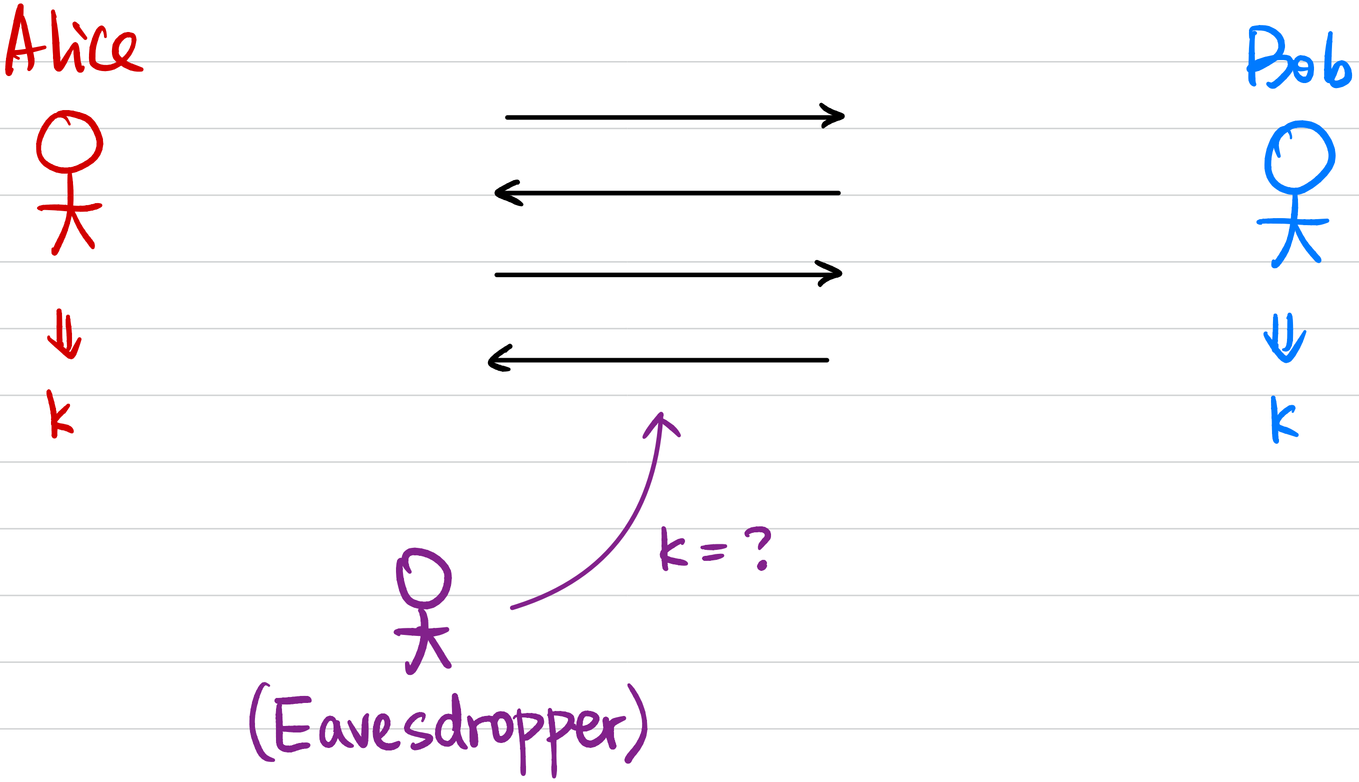
  $c = \langle c_1, c_2 \rangle$

  $m = c_2 \cdot (c_1^{sk})^{-1}$

Correctness?

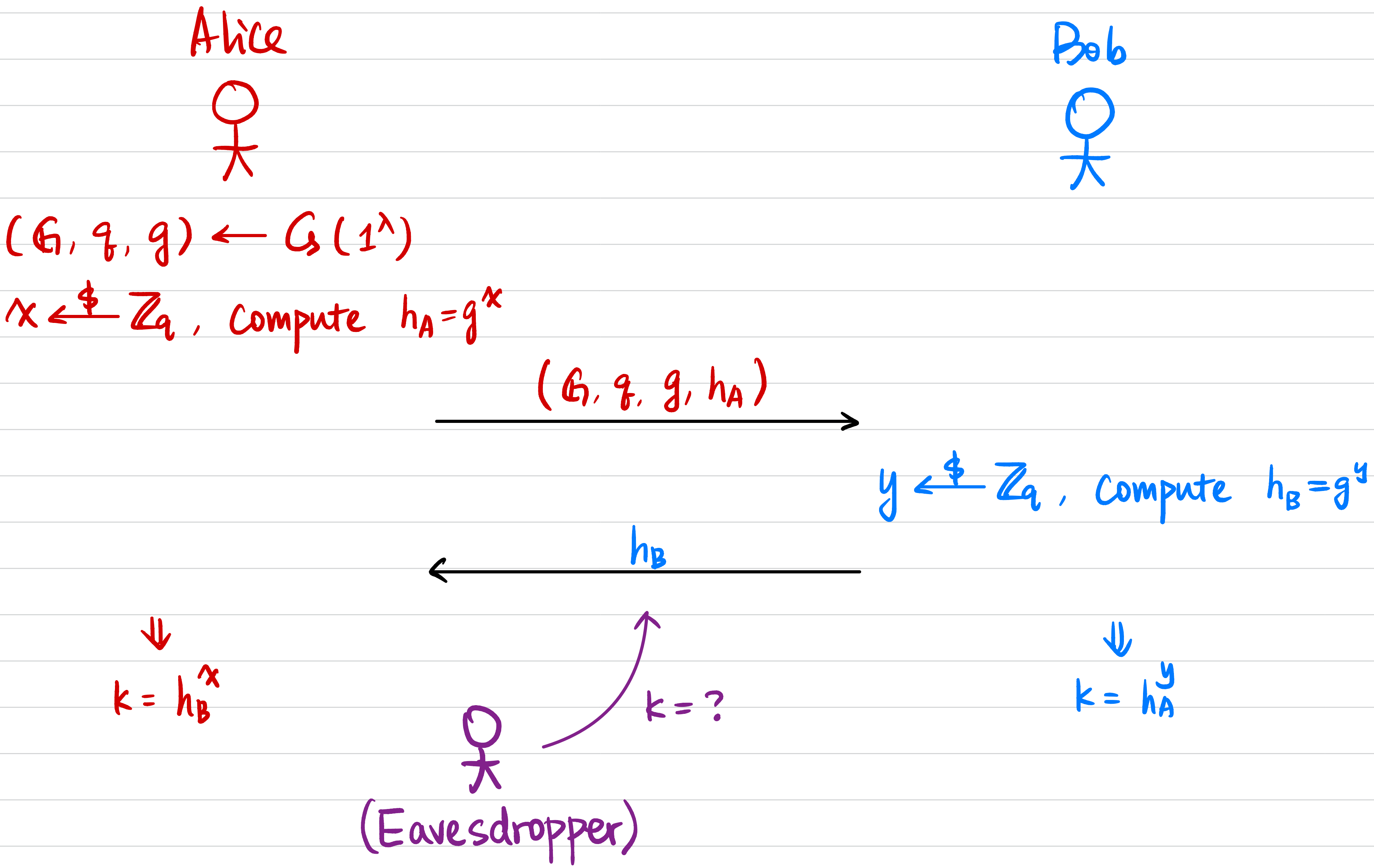
Security?

Secure Key Exchange



<u>Thm</u> (Informal): It's impossible to construct secure key exchang from SKE in a black-box way.

Diffie-Hellman Key Exchange



$(\mathbb{G}, q, g) \leftarrow \mathcal{G}(1^\lambda)$

$x \xleftarrow{\$} \mathbb{Z}_q$, compute $h_A = g^x$

$(\mathbb{G}, q, g, h_A)$

$y \xleftarrow{\$} \mathbb{Z}_q$, compute $h_B = g^y$

h_B

$\Downarrow$

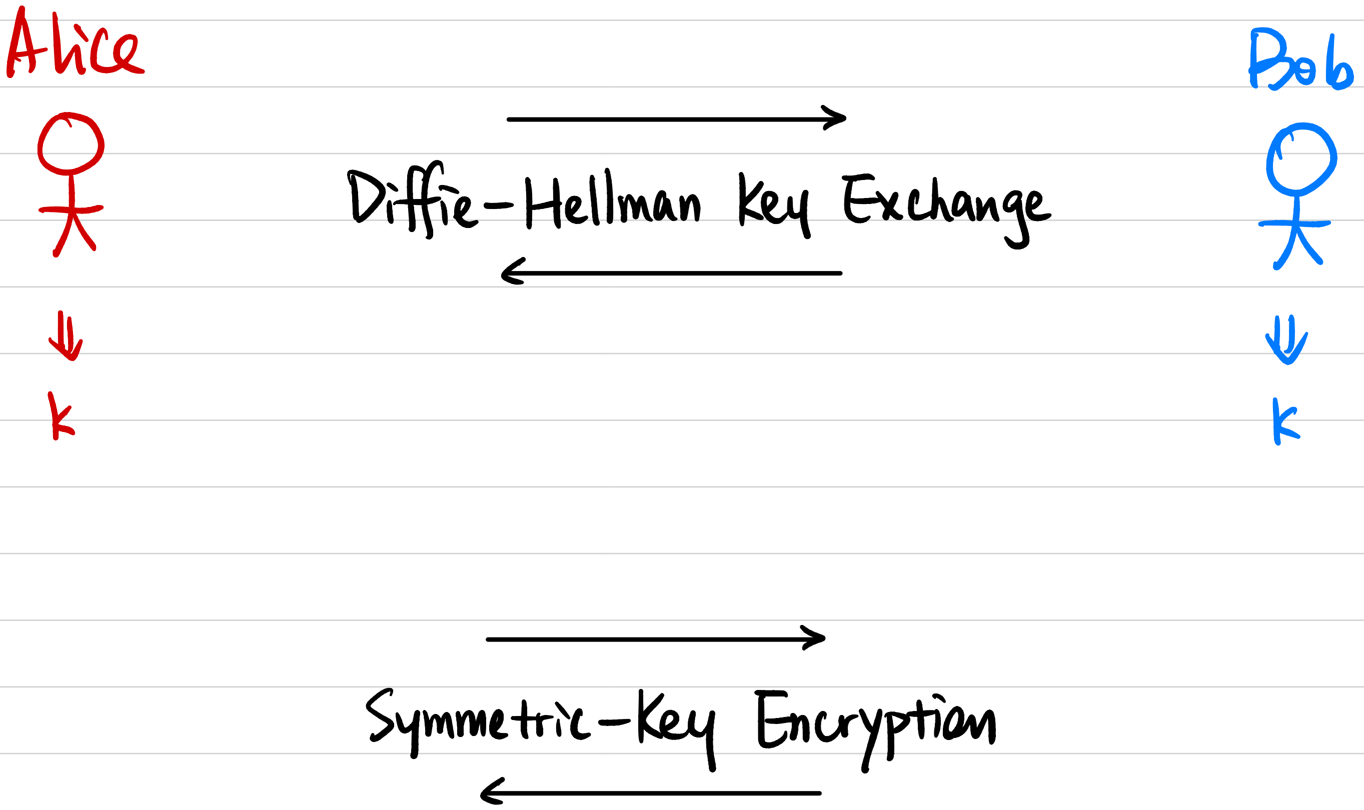
$k = h_B^x$

$k = ?$

$\Downarrow$

$k = h_A^y$

What happens in practice



Prime-Order Subgroups of $\mathbb{Z}_p^*$

Def For a group $(\mathbb{G}, \cdot)$, $\mathbb{H} \subseteq \mathbb{G}$ is a subgroup of $\mathbb{G}$ if $(\mathbb{H}, \cdot)$ forms a group.

A prime p is a safe prime if $p = 2q+1$ and q is a prime.

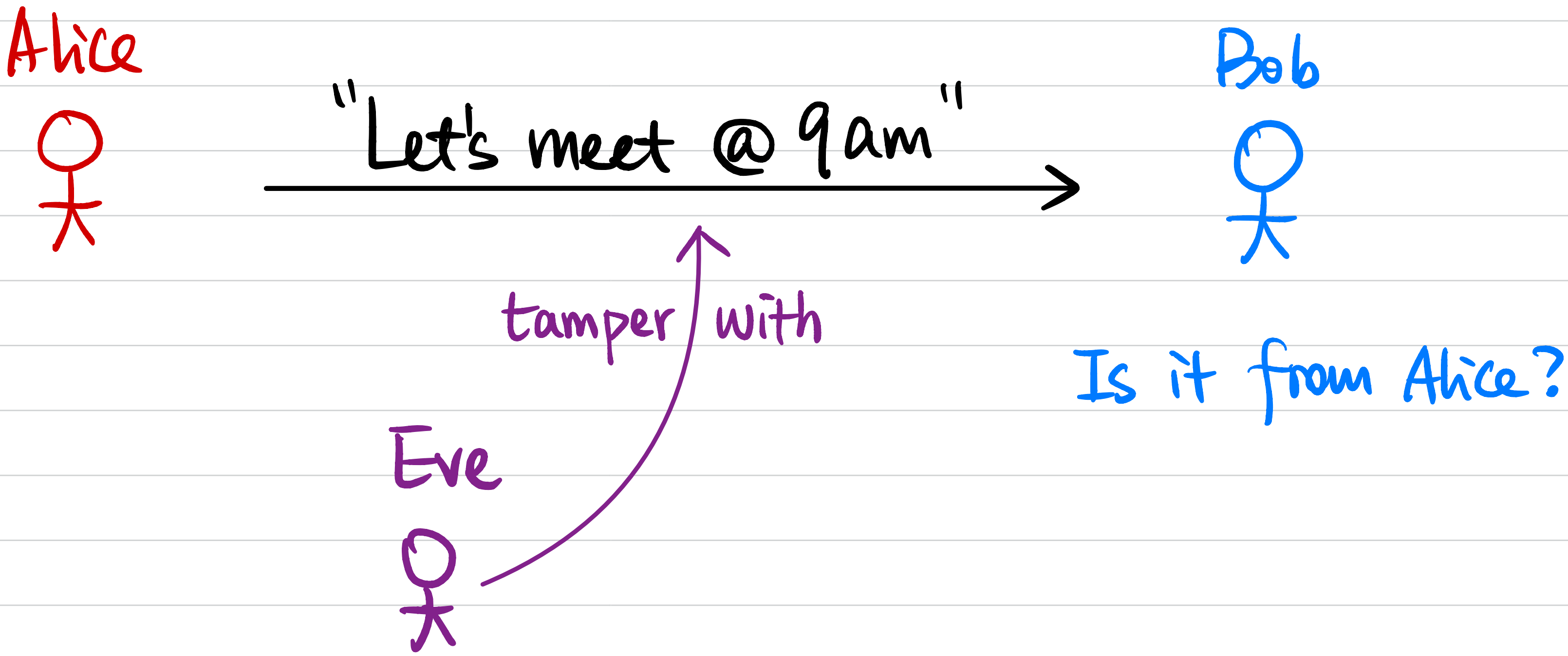
$\mathbb{Z}_p^*$ is a cyclic group of order $p-1 = 2q$.

Define $\mathbb{H} := \{ x^2 \bmod p \mid x \in \mathbb{Z}_p^* \}$

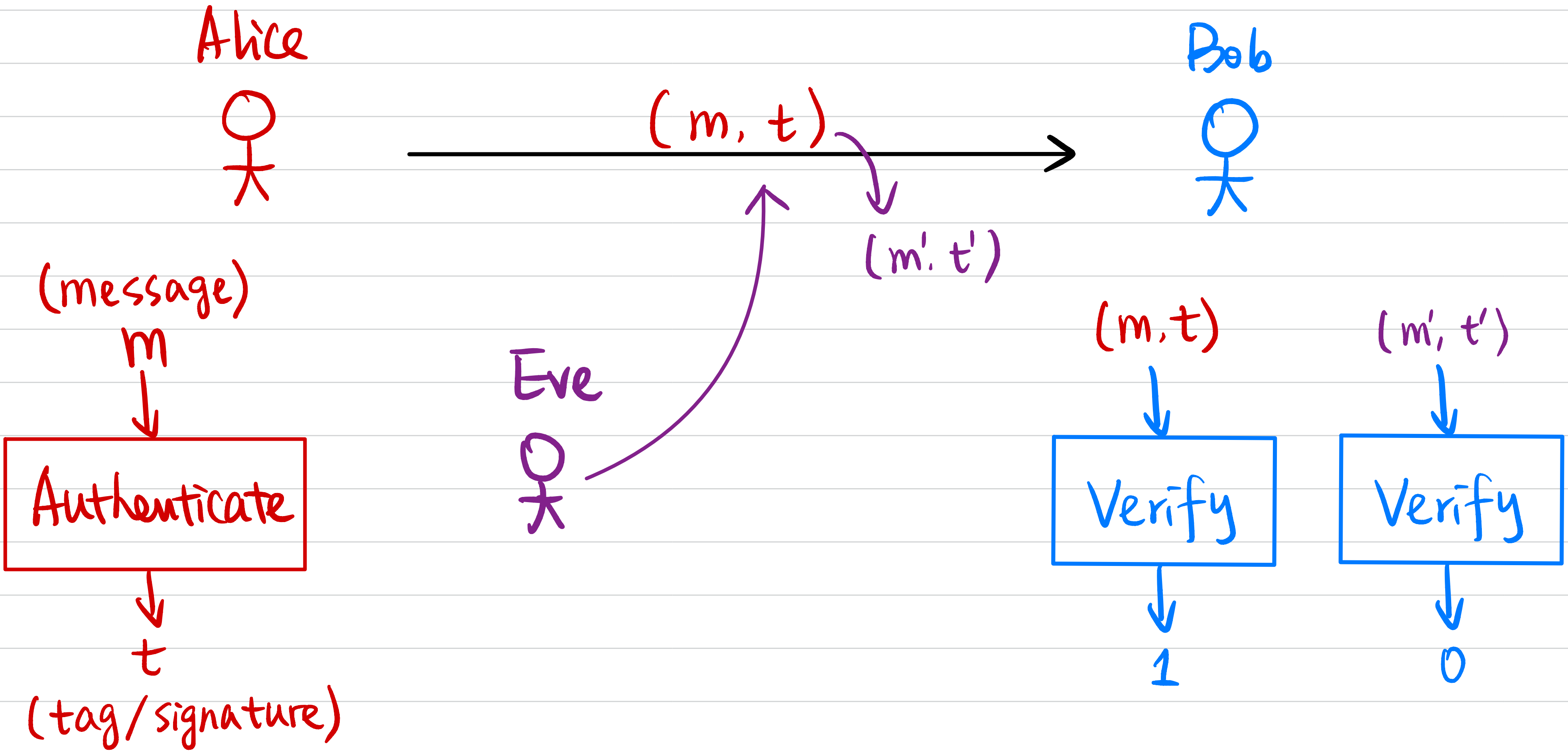
Thm $\mathbb{H}$ is a subgroup of $\mathbb{Z}_p^*$ of order q.

$p = 7$, $\mathbb{H} = \{1, 2, 4\}$

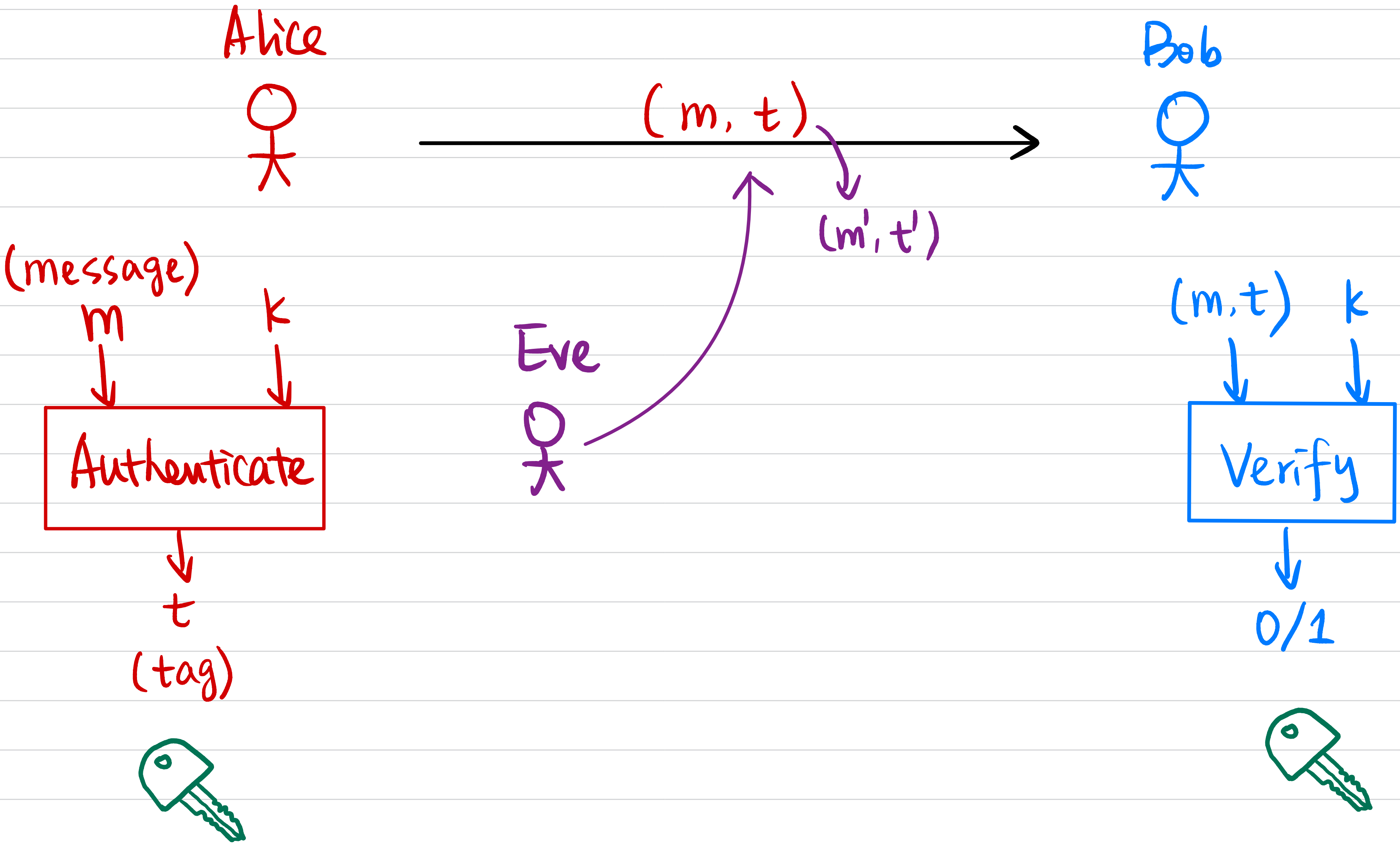
Message Integrity



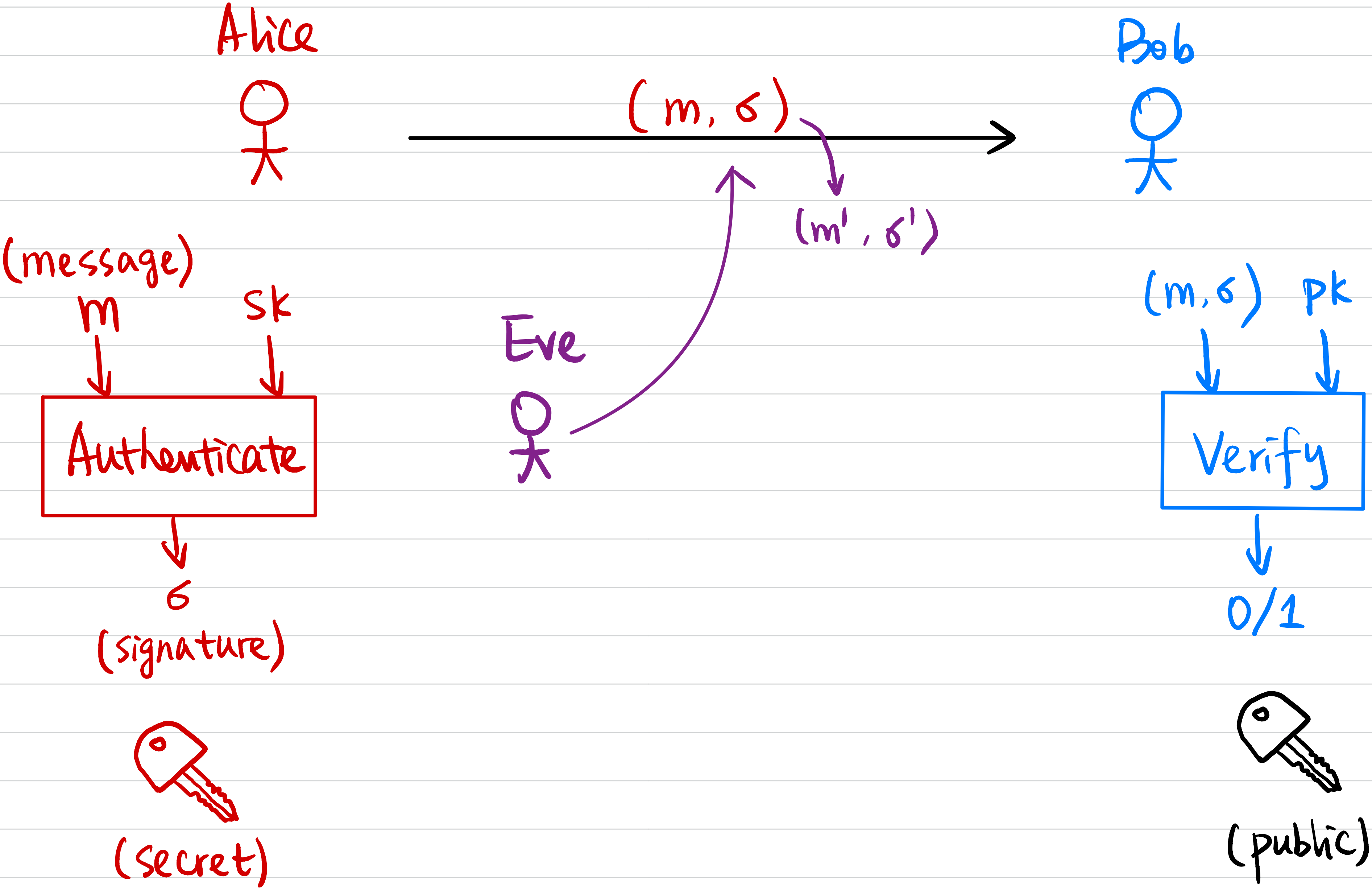
Message Integrity



Message Authentication Code (MAC)



Digital Signature



Syntax

Message Authentication Code (MAC) Scheme $\Pi = (\mathrm{Gen}, \mathrm{Mac}, \mathrm{Vrfy})$

$k \leftarrow \mathrm{Gen}(1^\lambda)$

$t \leftarrow \mathrm{Mac}_k(m)$

$0/1 := \mathrm{Vrfy}_k(m, t)$

Digital Signature Scheme $\Pi = (\mathrm{Gen}, \mathrm{Sign}, \mathrm{Vrfy})$

$(pk, sk) \leftarrow \mathrm{Gen}(1^\lambda)$

$\sigma \leftarrow \mathrm{Sign}_{sk}(m)$

$0/1 := \mathrm{Vrfy}_{pk}(m, \sigma)$

RSA Signature

Generate two n-bit primes p, q $(p \neq q)$

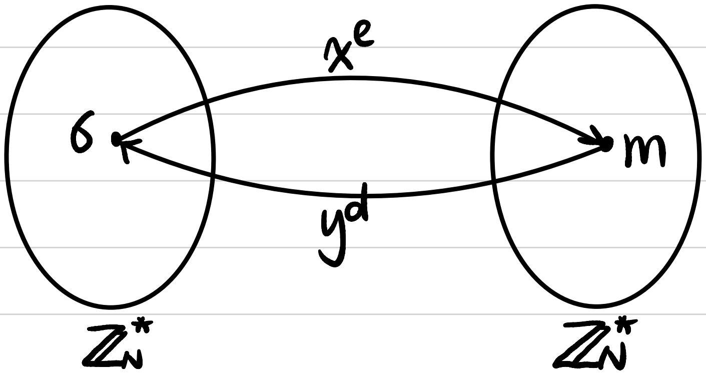
Compute $N = p \cdot q$, $\phi(N) = (p-1)(q-1)$

Choose e s.t. $\gcd(e, \phi(N)) = 1$

Compute $d = e^{-1} \bmod \phi(N)$.

Given (N, e) & a random $y \xleftarrow{\$} \mathbb{Z}_N^*$, it's computationally hard to find x s.t.

$$x^e \equiv y \pmod N$$



$sk = d$ $pk = (N, e)$

$\mathrm{Sign}_{sk}(m) = m^d \bmod N$

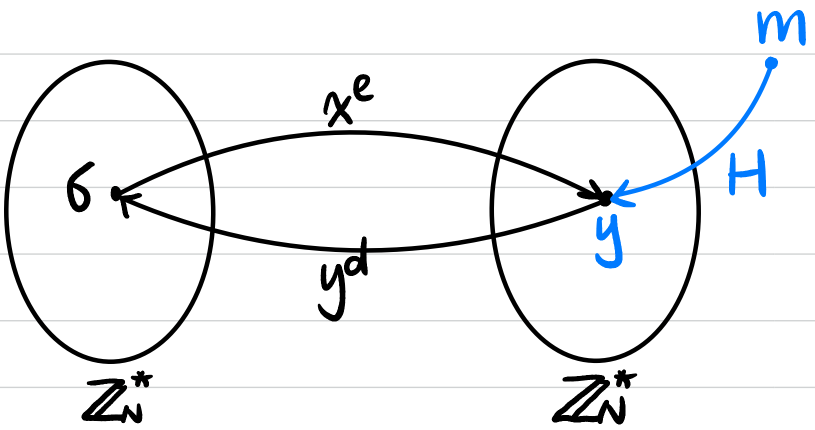
$\mathrm{Vrfy}_{pk}(m, \sigma)$: $\sigma^e \overset{?}{\equiv} m \pmod N$

Any security issue?

RSA Signature

Given (N,e) & a random $y \xleftarrow{\$} \mathbb{Z}_N^*$, it's computationally hard to find x s.t.

$$x^e \equiv y \pmod N$$



$sk = d$ $pk = (N, e)$

$\mathrm{Sign}_{sk}(m) = H(m)^d \bmod N$

$\mathrm{Vrfy}_{pk}(m, \sigma)$: $\sigma^e \overset{?}{\equiv} H(m) \pmod N$

$H: \{0,1\}^* \to \mathbb{Z}_N^*$

$\uparrow$

public, deterministic function that gives a (pseudo)random output

Security?