CSCI 1515 Applied Cryptography

This Lecture:

- SWHE from LWE (GSW, Continued)
- SWHE from RLWE (BFV)
- Private Information Retrieval (PIR)

FHE Constructions

Step 1: Somewhat Homomorphic Encryption (SWHE)

- over Integers
- from LWE (GSW)
- from RLWE (BFV)

Step 2: Bootstrapping

Learning With Errors (LWE) Assumption

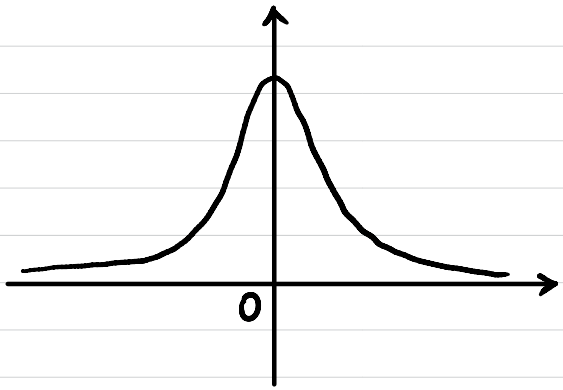
$n \sim$ security parameter

$q \sim 2^{n^\epsilon}$

$m = \Theta(n \log q)$

χ: distribution over $\mathbb{Z}_q$

(concentrated on "small integers")

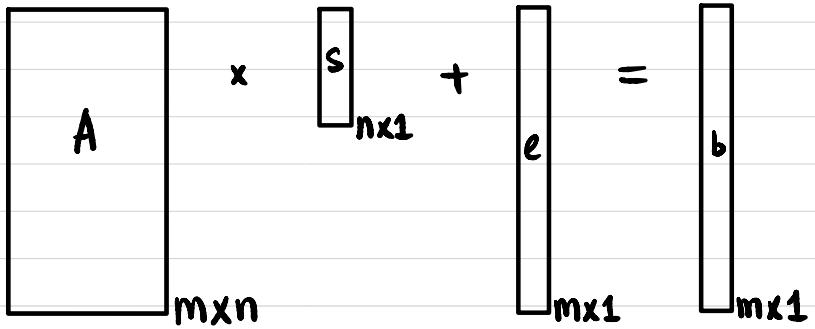


$$\Pr[|e| > \alpha \cdot q \mid e \leftarrow \chi] \leq \text{negl}(n)$$

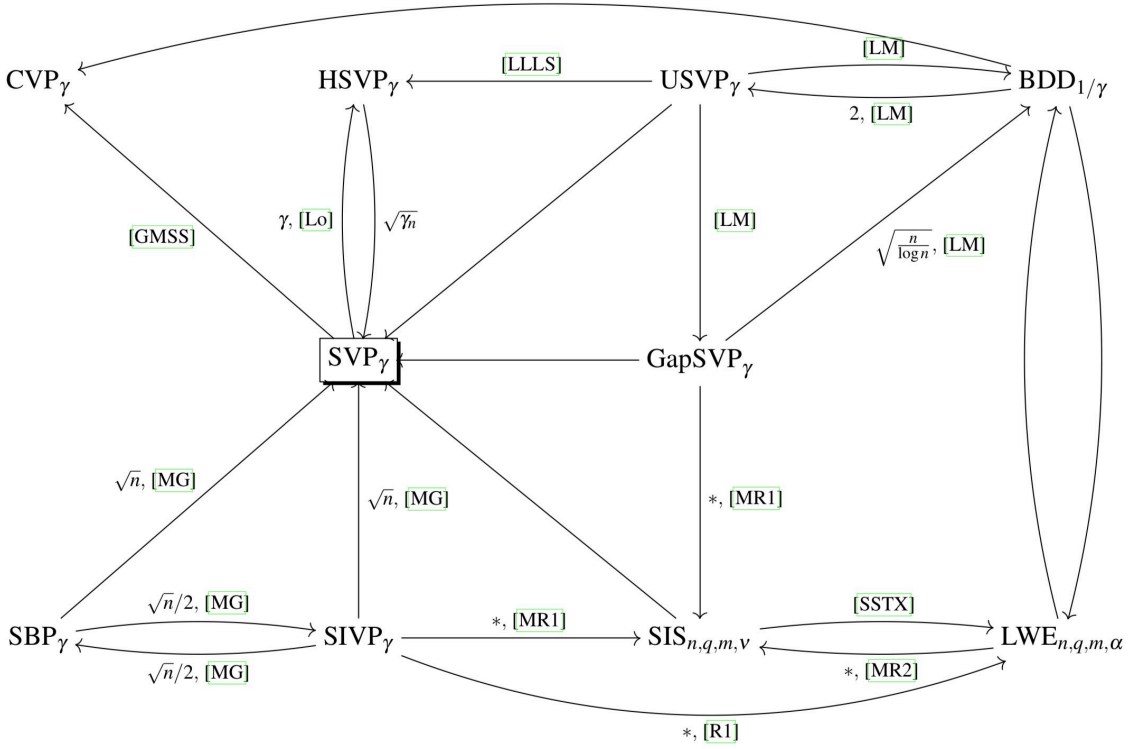
$\alpha \ll 1$

LWE$[n, m, q, \chi]$:

$A \xleftarrow{\$} \mathbb{Z}_q^{m \times n} \quad s \xleftarrow{\$} \mathbb{Z}_q^{n} \quad e \leftarrow \chi^m$



$$(A,\ b = As + e) \overset{c}{\approx} (A,\ b' \xleftarrow{\$} \mathbb{Z}_q^m)$$

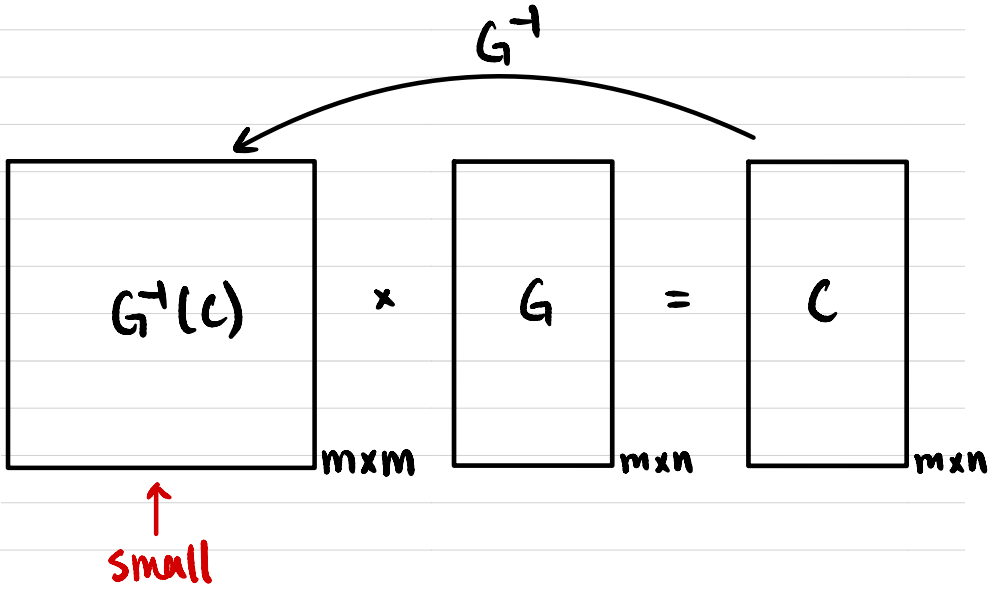

CVP_γ
HSVP_γ
[LLLS]
USVP_γ
[LM]
$\mathrm{BDD}_{1/\gamma}$
2, [LM]
[GMSS]
γ, [Lo]
$\sqrt{\gamma_n}$
[LM]
$\sqrt{\frac{n}{\log n}}$, [LM]
SVP_γ
GapSVP_γ
$\sqrt{n}$, [MG]
$\sqrt{n}$, [MG]
$*$, [MR1]
$\sqrt{n}/2$, [MG]
SBP_γ
SIVP_γ
$*$, [MR1]
$\mathrm{SIS}_{n,q,m,\nu}$
[SSTX]
$\mathrm{LWE}_{n,q,m,\alpha}$
$\sqrt{n}/2$, [MG]
$*$, [MR2]
$*$, [R1]

SWHE from LWE (GSW)

Attempt 2 (secret-key)

Flattering Gadget:

Gadget matrix $G \in \mathbb{Z}_q^{m \times n}$

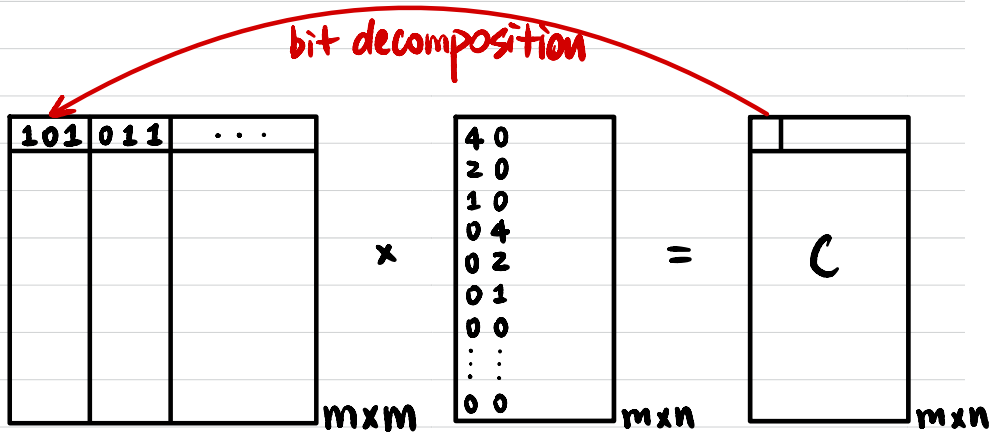


Inverse transformation

$$G^{-1}: \mathbb{Z}_q^{m \times n} \to \mathbb{Z}_q^{m \times m}$$

$\forall C \in \mathbb{Z}_q^{m \times n}$, $G^{-1}(C) = \text{small}$

$G^{-1}(C) \cdot G = C$



$m = n \cdot \log q$

SWHE from LWE (GSW)

Attempt 2 (secret-key)

$$= t_{n\times 1} \begin{bmatrix} s \\ 1 \end{bmatrix}_{n\times 1}$$

$Enc_{sk}(\mu)$: $\mu \in \{0,1\}$

Sample $C_0 \in \mathbb{Z}_q^{m\times n}$ s.t. $C_0 \cdot \vec{t} = \text{small}$

$$\begin{bmatrix} C_0 \end{bmatrix}_{m\times n} \times \begin{bmatrix} t \end{bmatrix}_{n\times 1} = \begin{bmatrix} e \end{bmatrix}_{m\times 1}$$

$$C = C_0 + \mu \cdot G$$

↑ gadget matrix

$Dec_{sk}(C)$: $C \cdot \vec{t} = (C_0 + \mu \cdot G) \cdot \vec{t}$

$= \vec{e} + \mu \cdot (G \cdot \vec{t})$

Homomorphism: $C_1 \cdot \vec{t} = \mu_1 \cdot (G \cdot \vec{t}) + \vec{e}_1$

$C_2 \cdot \vec{t} = \mu_2 \cdot (G \cdot \vec{t}) + \vec{e}_2$

Additive Homomorphism?

$C = C_1 + C_2 \Rightarrow C \cdot \vec{t} = (\mu_1 + \mu_2) \cdot (G \cdot \vec{t}) + (\vec{e}_1 + \vec{e}_2)$

Multiplicative Homomorphism?

$C = G^{-1}(C_1) \cdot C_2$

$C \cdot \vec{t} = G^{-1}(C_1) \cdot C_2 \cdot \vec{t}$

$= G^{-1}(C_1) \cdot (\mu_2 \cdot (G \cdot \vec{t}) + \vec{e}_2)$

$= \mu_2 \cdot G^{-1}(C_1) \cdot G \cdot \vec{t} + G^{-1}(C_1) \cdot \vec{e}_2$

$= \mu_2 \cdot C_1 \cdot \vec{t} + G^{-1}(C_1) \cdot \vec{e}_2$

$= \mu_2 \cdot (\mu_1 \cdot (G \cdot \vec{t}) + \vec{e}_1) + G^{-1}(C_1) \cdot \vec{e}_2$

$= \mu_2 \cdot \mu_1 \cdot (G \cdot \vec{t}) + \mu_2 \cdot \vec{e}_1 + G^{-1}(C_1) \cdot \vec{e}_2$

How homomorphic is it?

$\#\text{MULT} \sim \lg_m q$

Ring LWE (RLWE) Assumption

Polynomial ring $R = \mathbb{Z}[x] / (x^m + 1)$ where $m = 2^k$

↑ polynomials with integer coefficients modulo $(x^m + 1)$

$$R_q = \mathbb{Z}_q[x] / (x^m + 1)$$

↑ polynomials with integer coefficients modulo q and $(x^m + 1)$

χ : "noise" distribution over R

$a \xleftarrow{\$} R_q \qquad s \xleftarrow{\$} R_q \text{ (or } s \leftarrow \chi) \qquad e \leftarrow \chi$

$$(a, [a \cdot s + e]_q) \overset{c}{\simeq} (a, b \xleftarrow{\$} R_q)$$

SWHE from RLWE (BFV)

Plaintext space $R_t = \mathbb{Z}_t[x] / (x^m + 1)$

Ciphertext space $R_q \times R_q$

$$\Delta := \left\lfloor \frac{q}{t} \right\rfloor$$

$a \xleftarrow{\$} R_q \quad s \leftarrow \chi \quad e \leftarrow \chi$

$$pk = \left([-(a \cdot s + e)]_q, \; a \right)$$

$$sk = s$$

$Enc_{pk}(m)$: $m \in R_t$

Sample $u, e_1, e_2 \leftarrow \chi$

$$c = \left([pk_0 \cdot u + e_1 + \Delta \cdot m]_q, \; [pk_1 \cdot u + e_2]_q \right)$$

$Dec_{sk}(c)$: $[c_0 + c_1 \cdot s]_q = ?$

SWHE from RLWE (BFV)

$$[C(s)]_q = c_0 + c_1 \cdot s = \Delta \cdot m + e$$

Homomorphism:

$$[C^{(1)}(s)]_q = \Delta \cdot m_1 + e_1$$

$$[C^{(2)}(s)]_q = \Delta \cdot m_2 + e_2$$

Additive Homomorphism?

Multiplicative Homomorphism?

SWHE from RLWE (BFV)

$$[C(s)]_q = c_0 + c_1 \cdot s + c_2 \cdot s^2 = \Delta \cdot m + e$$

$$\downarrow$$

$$[c'(s)]_q = c_0' + c_1' \cdot s = \Delta \cdot m + e$$

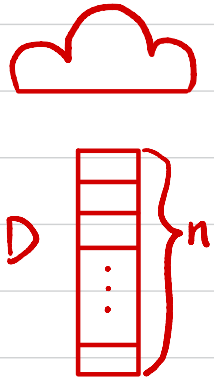
Relinearization:

Relinearization key: $rlk = \left([-(a \cdot s + e + s^2)]_q,\ a\right)$

$$[rlk(s)]_q = -s^2 + \text{small}$$

$c(s) + c_3 \cdot rlk(s)$?

Private Information Retrieval (PIR)



WANT: D[i]

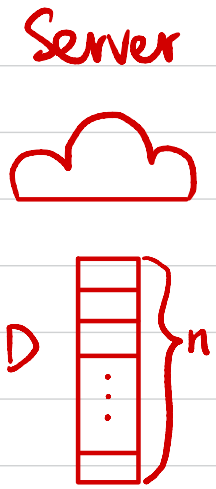
While hiding i against Server

Trivial Solution:

Communication complexity $O(n)$

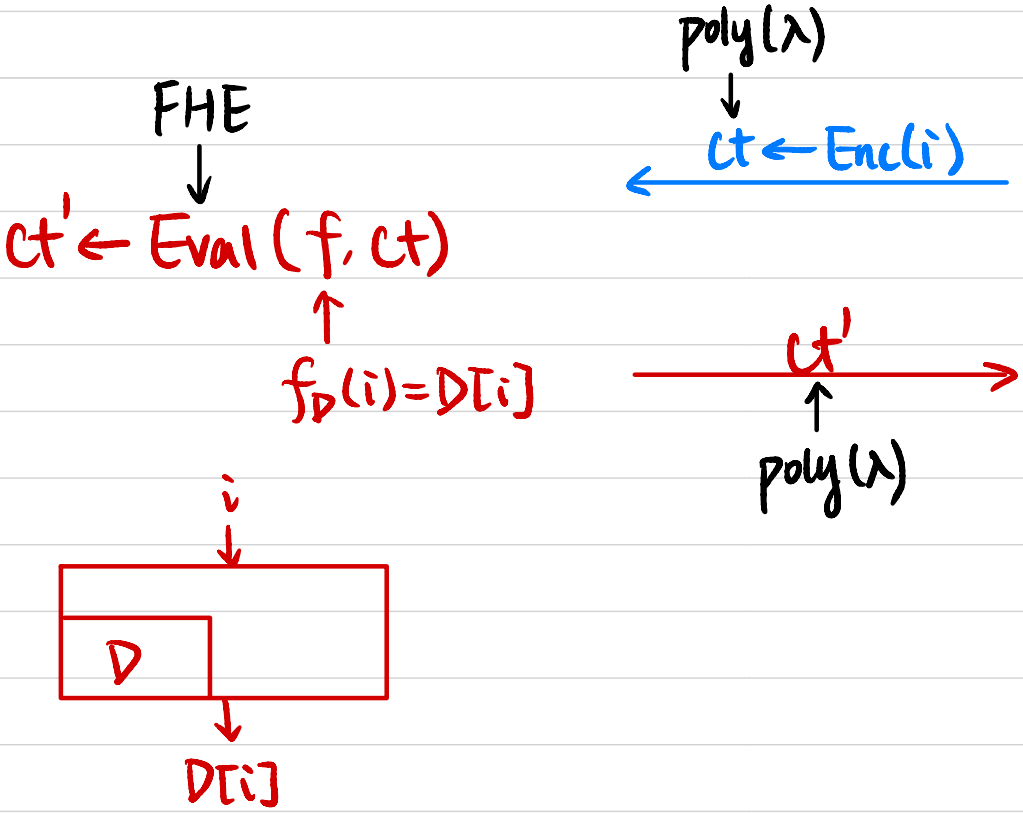
Goal: Communication complexity $o(n)$

Private Information Retrieval (PIR)



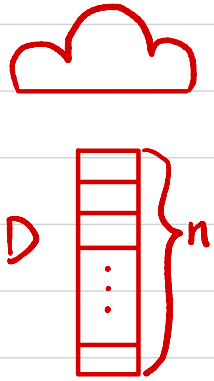
WANT: D[i]

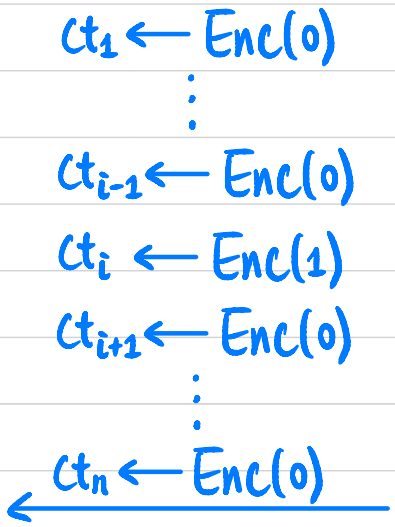
While hiding i against Server

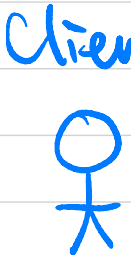


D[i] = Dec(ct')

Private Information Retrieval (PIR)







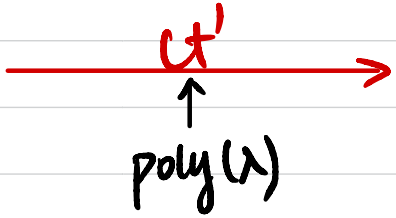
WANT: D[i]

While hiding i against Server

Homomorphic Scalar Mult

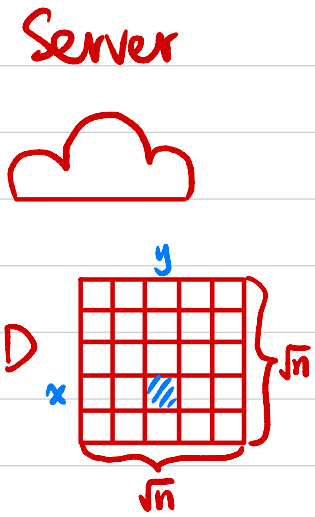
$$ct' \leftarrow \sum_{i=1}^{n} D[i] \cdot ct_i$$

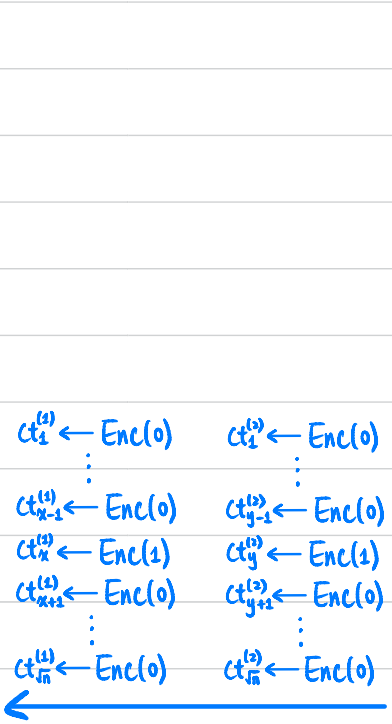
Homomorphic Add

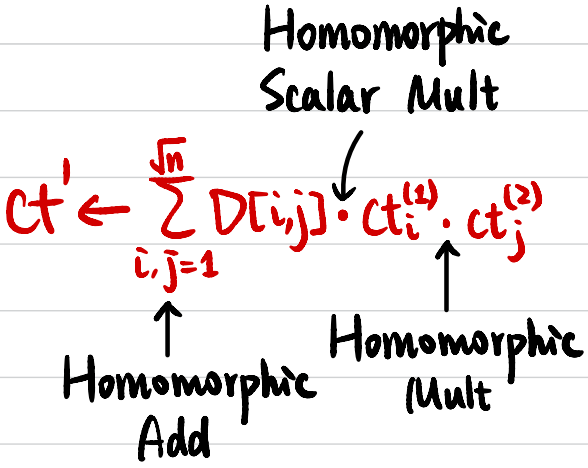


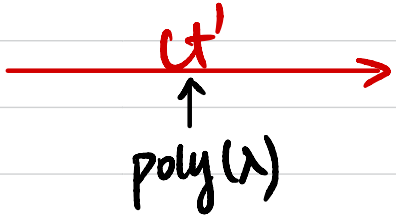
$D[i] = Dec(ct')$

Private Information Retrieval (PIR)





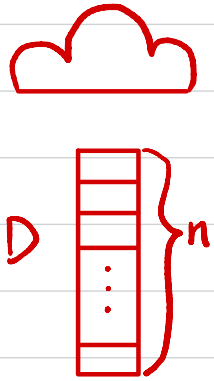




$D[x,y] = Dec(ct')$

Extend to dimension d ?

PIR from GSW



WANT: $D[i]$

While hiding i against Server

$\overline{i = b_1 b_2 \cdots b_\ell}$ ($\log n$)

$\forall i \in [n]$:

$\overline{i = b'_1 b'_2 \cdots b'_\ell}$

$ct_1 \leftarrow Enc(b_1)$

$\vdots$

$ct_\ell \leftarrow Enc(b_\ell)$

Homomorphic Scalar Mult

$ct'_i \leftarrow D[i] \cdot \prod_{t=1}^{\ell} Enc(b_t \stackrel{?}{=} b'_t)$

Homomorphic Mult

ct'

poly(λ)

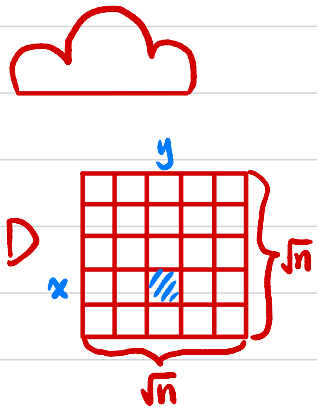
$ct' \leftarrow \sum_{i=1}^{n} ct'_i$

Homomorphic Add

$D[i] = Dec(ct')$

PIR from Additive HE

Server



Client

$ct_1 \leftarrow Enc(0)$

$\vdots$

$ct_{x-1} \leftarrow Enc(0)$

$ct_x \leftarrow Enc(1)$

$ct_{x+1} \leftarrow Enc(0)$

$\vdots$

$ct_{\sqrt{n}} \leftarrow Enc(0)$

WANT: $D[x,y]$

While hiding (x,y) against Server

Homomorphic Scalar Mult

$$\forall j \in [\sqrt{n}],\ ct'_j \leftarrow \sum_{i=1}^{\sqrt{n}} D[i,j] \cdot ct_i$$

Homomorphic Add

$ct'_1, \cdots, ct'_{\sqrt{n}}$

$D[x,y] = ?$

Application: Secure 2PC ?

Alice

Bob

$C(x, y)$

Input: x

Input: y

$ct \leftarrow Enc(y)$

$ct' \leftarrow Eval(f, ct)$

$f_x(y) = C(x, y)$

$f(y) \leftarrow Dec_{sk}(ct')$