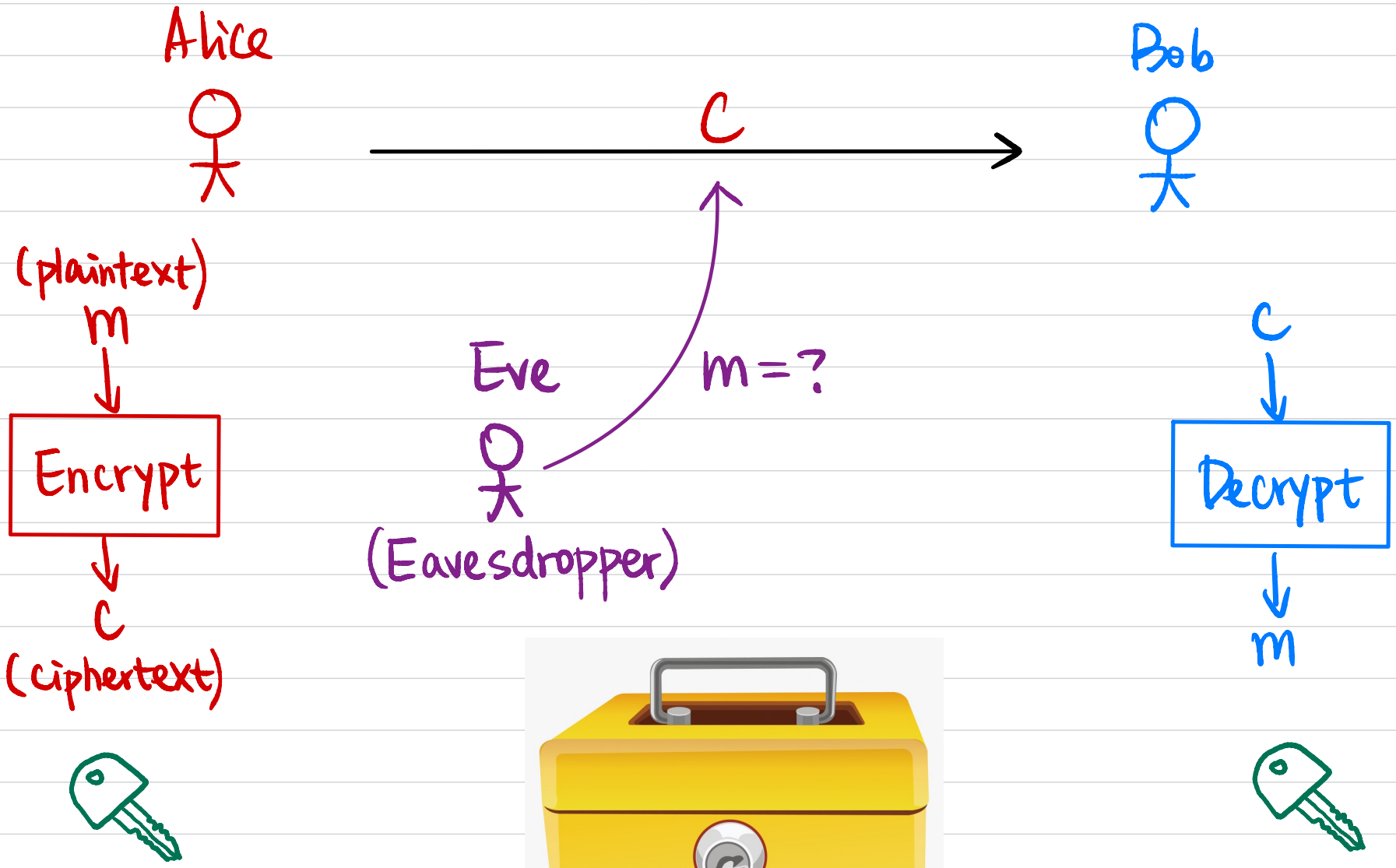
CSCI 1510

This Lecture:

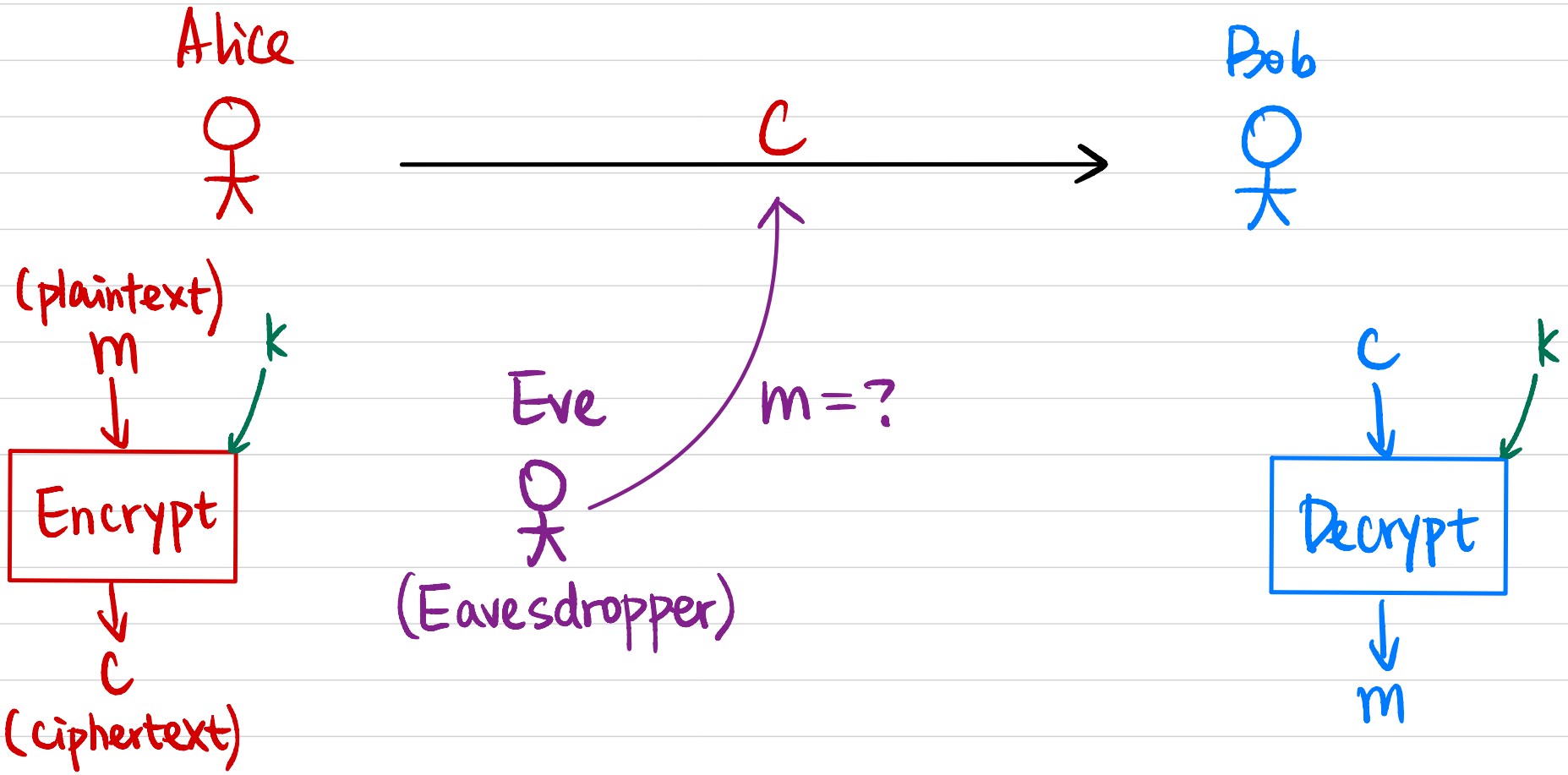
- Syntax of Symmetric-Key Encryption
- Kerckhoff's Principle
- Definition of Perfect Security
- One-Time Pad
- Limitations of Perfect Security

Message Secrecy



Symmetric-Key Encryption

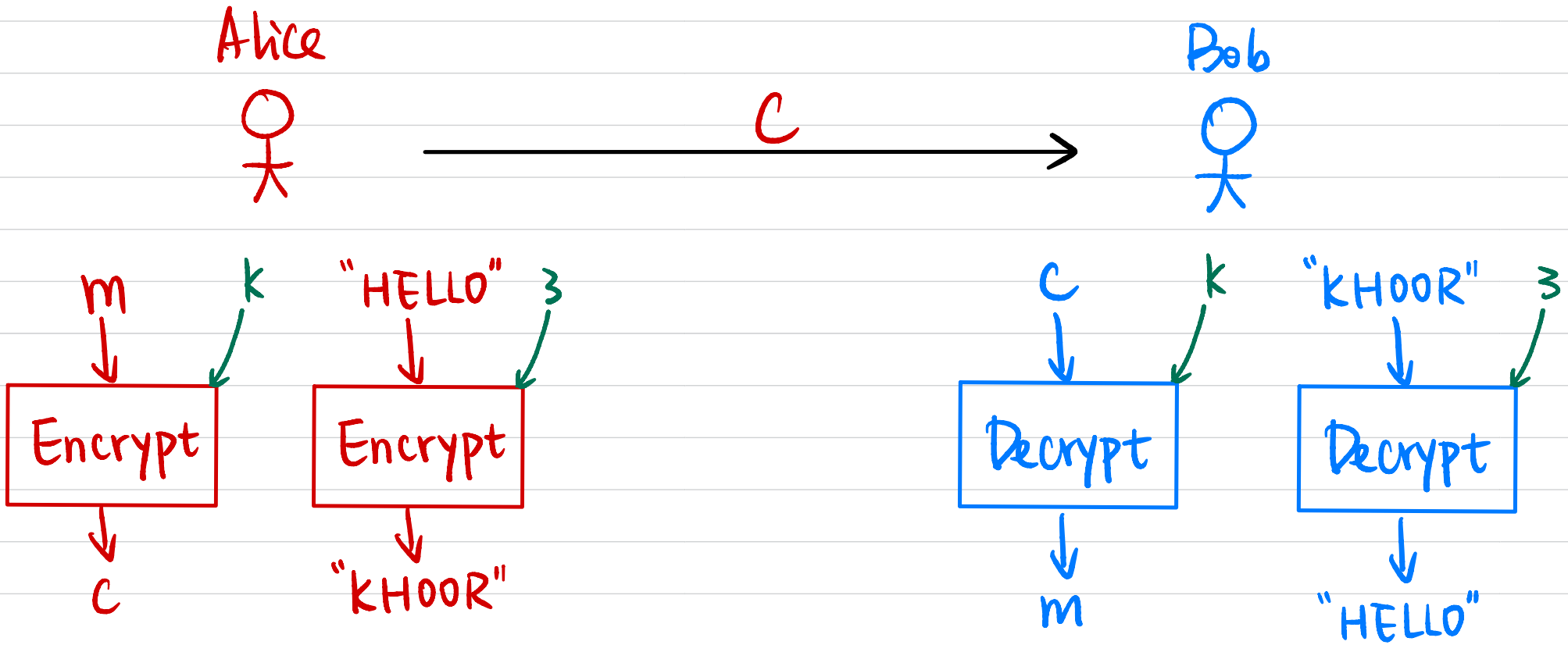
Private-Key / Secret-Key



How to define security ?

Example: Shift Cipher

$k \xleftarrow{\$} \{0, 1, \cdots, 25\}$



Shift each character forward by k.

How?

Symmetric-Key Encryption

Private-Key / Secret-Key

- **Syntax:**

  A **symmetric-key encryption scheme** is defined by a message space $\mathcal{M}$, a key space $\mathcal{K}$, and algorithms (Gen, Enc, Dec):

  $k \leftarrow Gen$

  $c \leftarrow Enc(k, m)$ $\quad Enc_k(m)$

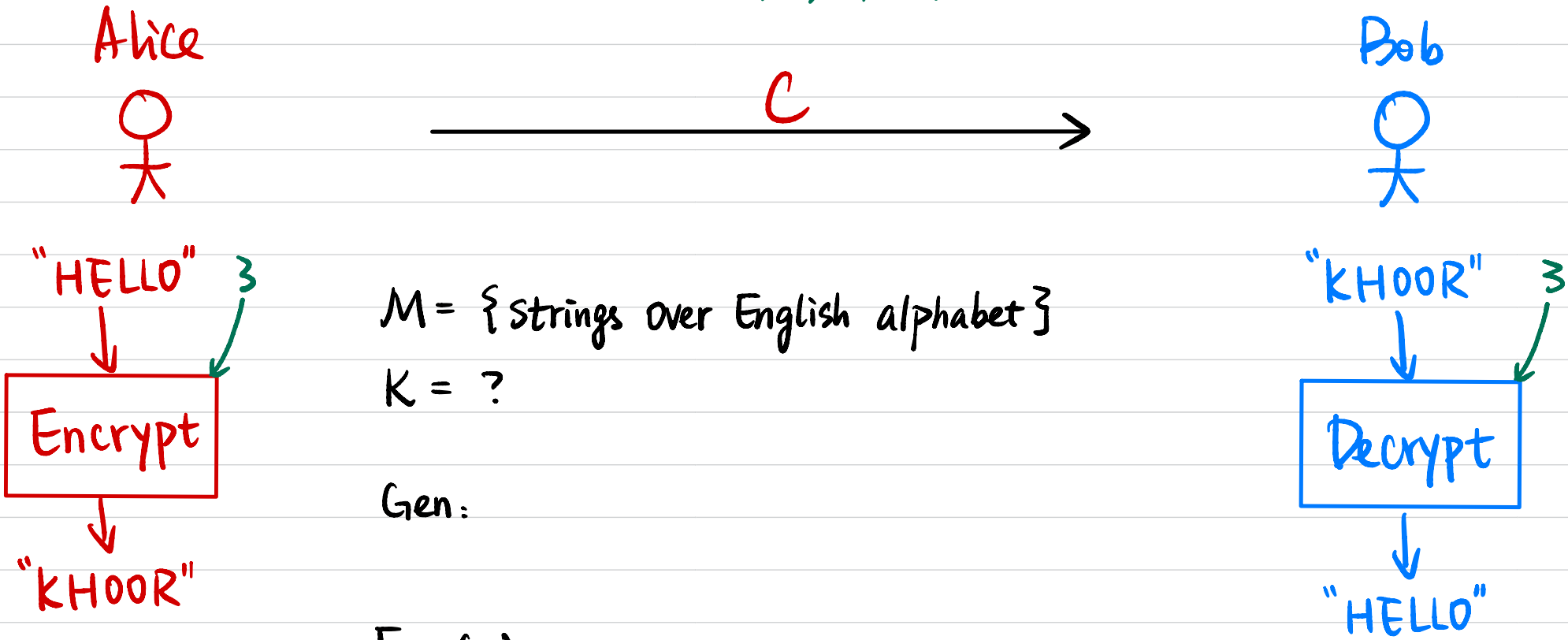
  $m/\perp := Dec(k, c)$ $\quad Dec_k(c)$

- **Correctness:** $\forall m \in \mathcal{M}$, $\forall k$ output by Gen,

  $$Dec_k(Enc_k(m)) = m$$

Example: Shift Cipher

$k \xleftarrow{\$} \{0, 1, \cdots, 25\}$



$M = \{\text{strings over English alphabet}\}$

$K = ?$

Gen:

$Enc_k(m)$: $m = m_1 m_2 \cdots m_L$

$Dec_k(c)$: $c = c_1 c_2 \cdots c_L$

Symmetric-Key Encryption

Private-Key / Secret-Key

- **Syntax:**

A **symmetric-key encryption scheme** is defined by a message space $\mathcal{M}$, a key space $\mathcal{K}$, and algorithms (Gen, Enc, Dec):

$k \leftarrow Gen$

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k must be kept secret

Keep (Gen, Enc, Dec) secret as well?

- **Correctness:** $\forall m \in \mathcal{M}$, $\forall k$ output by Gen,

$$Dec_k(Enc_k(m)) = m$$

Kerckhoff's Principle

The cipher method must not be required to be secret, and it must be able to fall into the hands of the enemy without inconvenience.

↑ Only the key is kept secret

Why ?

How to define security?

- It's impossible for Eve to recover k from c.

$Enc_k(m) = m$

- It's impossible for Eve to recover m from c.

90% of m?

- It's impossible for Eve to recover any character of m from c.

Distribution of m?

Already knows some characters of m?

Perfect Security

Regardless of any information an attacker already has,

a ciphertext should leak no additional information about the plaintext.

Notation

$\mathcal{K}$: key space

$\mathcal{M}$: message / plaintext space

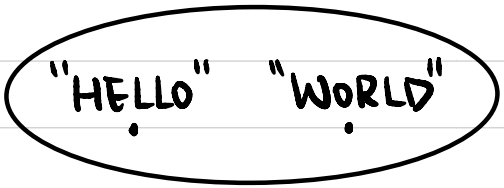
$\mathcal{C}$: ciphertext space

K: random variable denoting the output of Gen.

$$\Pr[K = k] = \Pr[\text{Gen outputs } k].$$

M: random variable denoting the message / plaintext to be encrypted.

Example: $\mathcal{M} = \{\text{"HELLO"}, \text{"WORLD"}\}$



$\Pr[M = \text{"HELLO"}] = 0.3$

$\Pr[M = \text{"WORLD"}] = 0.7$

C: random variable denoting the resulting ciphertext.

① $k \leftarrow \text{Gen}$

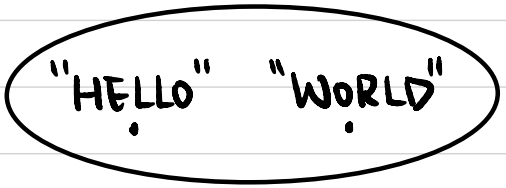
② $m \leftarrow \mathcal{M}$ (following a certain distribution)

③ $c \leftarrow \text{Enc}_k(m)$

Example: Shift Cipher

K: $Pr[K = k] = ?$

M: $M = \{\text{"HELLO"}, \text{"WORLD"}\}$



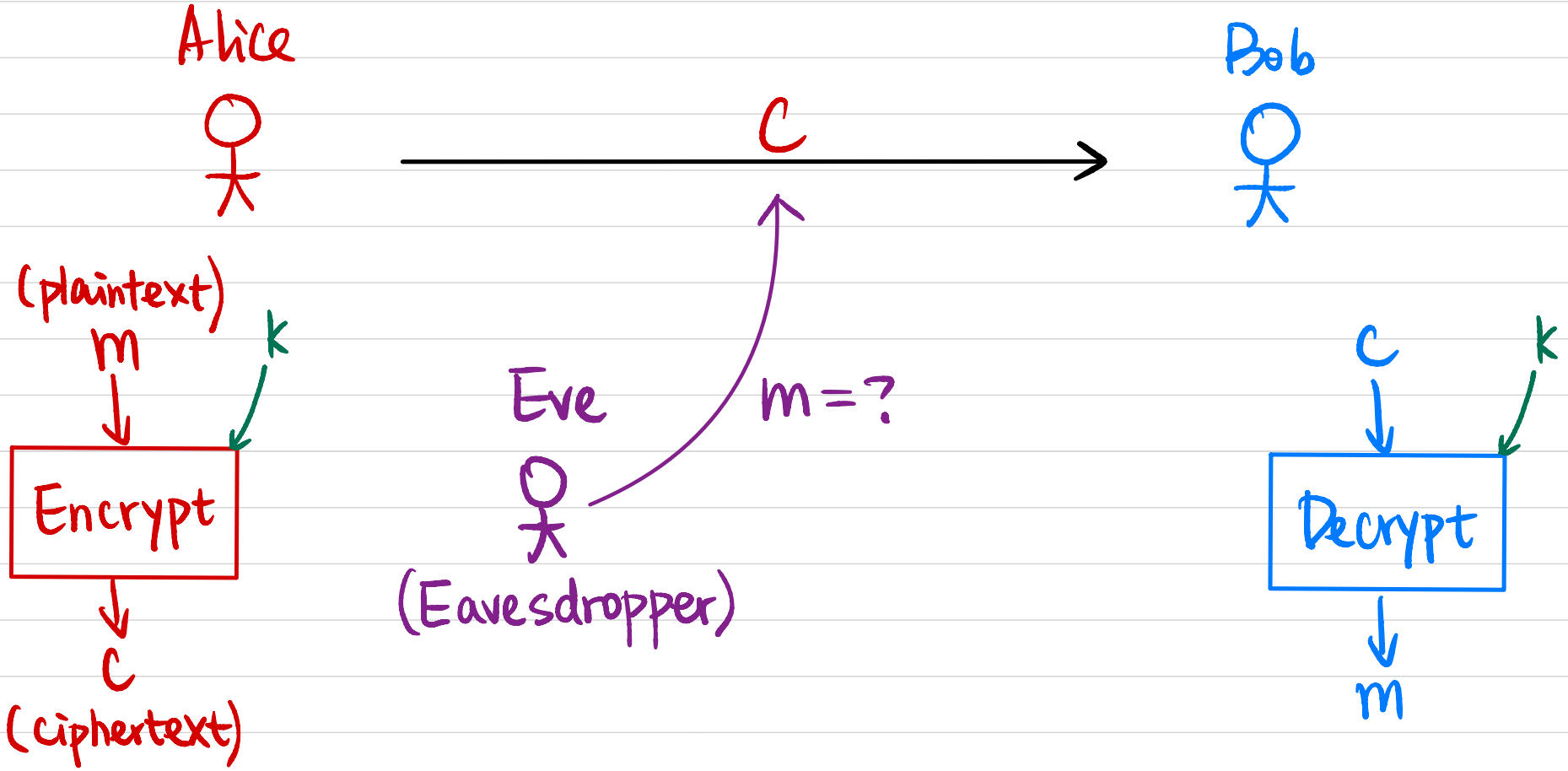
$Pr[M = \text{"HELLO"}] = 0.3$

$Pr[M = \text{"WORLD"}] = 0.7$

C: $Pr[C = c] = ?$

$Pr[C = \text{"KHOOR"}] = ?$

Symmetric-Key Encryption



Eve knows: ① K, M, C, (Gen, Enc, Dec)

② distribution over M

③ ciphertext c

Perfect Security

Def 1 A symmetric-key encryption scheme (Gen, Enc, Dec) with message space $\mathcal{M}$ is **perfectly secure** if

$\forall$ probability distribution over $\mathcal{M}$.

$\forall m \in \mathcal{M}$.

$\forall c \in \mathcal{C}$ for which $\Pr[C=c] > 0$:

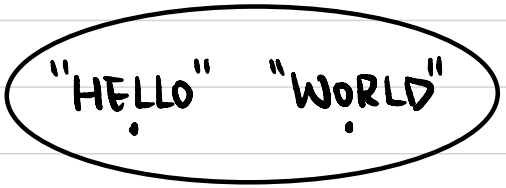
$$\Pr[M=m \mid C=c] = \Pr[M=m].$$

Example: Shift Cipher

$\Pr[M=m \mid C=c] \stackrel{?}{=} \Pr[M=m]$

K: $\Pr[K=k] = ?$

M: $M = \{\text{"HELLO"}, \text{"WORLD"}\}$



$\Pr[M=\text{"HELLO"}] = 0.3$

$\Pr[M=\text{"WORLD"}] = 0.7$

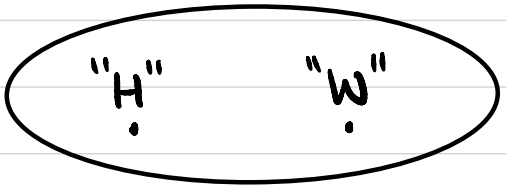
$\Pr[M=\text{"HELLO"} \mid C=\text{"KHOOR"}] = ?$

Example: Shift Cipher

$\Pr[M=m \mid C=c] \stackrel{?}{=} \Pr[M=m]$

K: $\Pr[K=k] = ?$

M: $M = \{\text{"H"}, \text{"W"}\}$



$\Pr[M=\text{"H"}] = 0.3$

$\Pr[M=\text{"W"}] = 0.7$

$\Pr[M=\text{"H"} \mid C=\text{"k"}] = ?$

Perfect Security

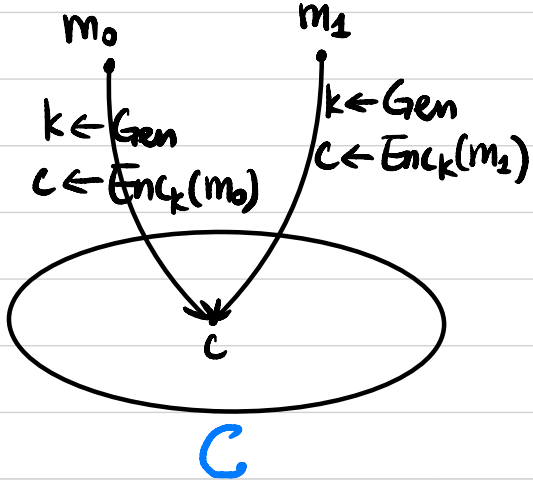
Def 2 A symmetric-key encryption scheme (Gen, Enc, Dec) with message space $\mathcal{M}$ is perfectly secure if

$\forall\, m_0, m_1 \in \mathcal{M}$,

$\forall\, c \in \mathcal{C}$:

$$\Pr[\mathrm{Enc}_k(m_0) = c] = \Pr[\mathrm{Enc}_k(m_1) = c]$$

over choice of k & randomness of Enc



Def 1 $\forall$ probability distribution over $\mathcal{M}$,

$\forall m \in \mathcal{M}$,

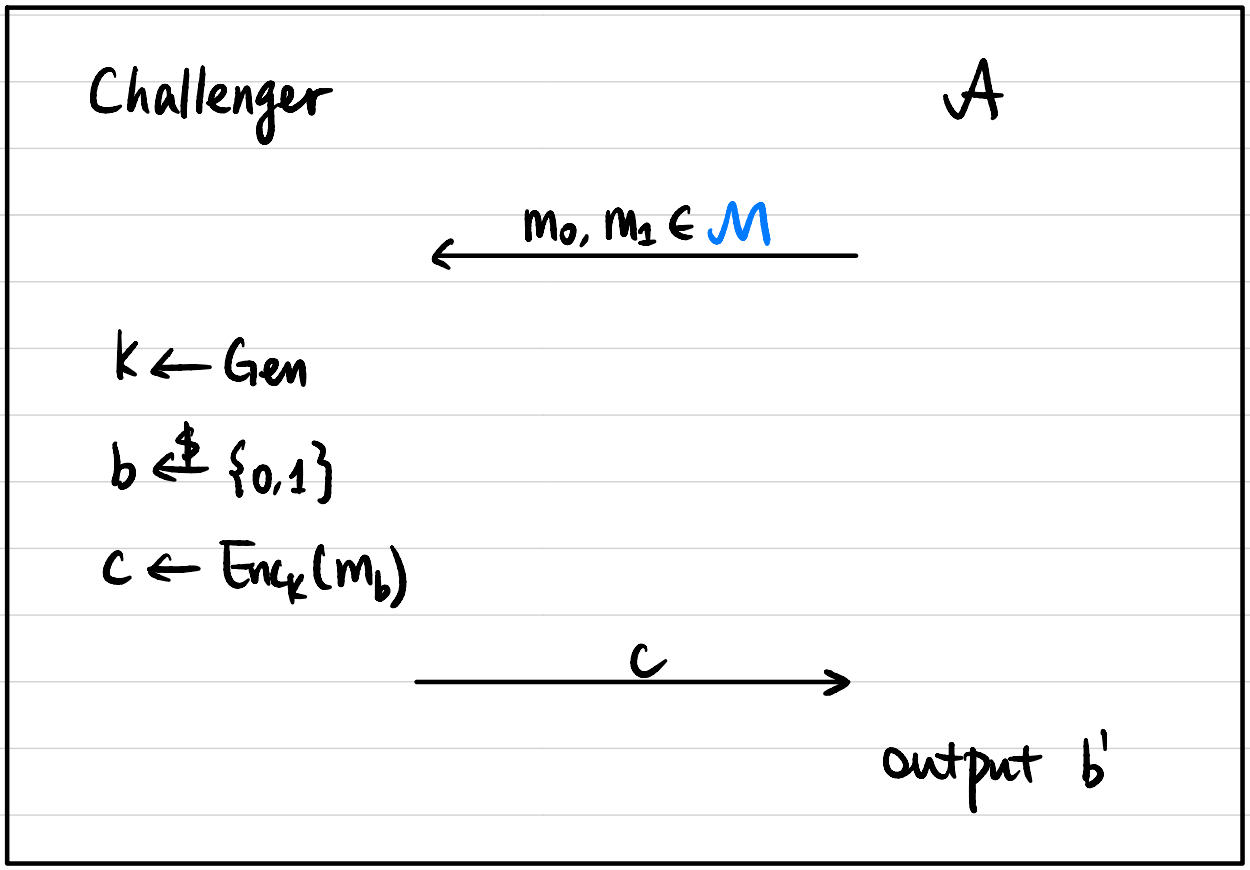
$\forall c \in \mathcal{C}$ for which $\Pr[C = c] > 0$:

$$\Pr[M = m \mid C = c] = \Pr[M = m].$$

Perfect Security

Def 3 (Game-based) A symmetric-key encryption scheme (Gen, Enc, Dec) with message space $\mathcal{M}$ is **perfectly indistinguishable** if $\forall \mathcal{A}$:

$$\Pr[b = b'] = \frac{1}{2}$$



One-Time Pad (OTP)

Fix an integer $\ell > 0$.

$K, M, C = \{0,1\}^\ell$ all ℓ-bit strings

- Gen: $k \xleftarrow{\$} \{0,1\}^\ell$, output k.
- $\text{Enc}_k(m)$: output $c := m \oplus k$
- $\text{Dec}_k(c)$: output $m := c \oplus k$

$\oplus$	0	1
0	0	1
1	1	0

Example: $\ell = 5$.

$k = 01101$

Enc: $m = 00110$

$c = 01011$

Dec: $k = 01101$

$m = 00110$

- Correctness?

- Security? $\forall\, m_0, m_1 \in M,\ \forall\, c \in C$:

$$\Pr[\text{Enc}_K(m_0) = c] = \Pr[C = c \mid M = m_0] = ?$$

$$\Pr[\text{Enc}_K(m_1) = c] = \Pr[C = c \mid M = m_1] = ?$$

One-Time Pad (OTP)

Limitations:

① Key is as long as the plaintext

② Cannot reuse the key ← why?

Can we make $|M| > |K|$?

Limitations of Perfect Security

Thm If $\Pi = (Gen, Enc, Dec)$ is a perfectly secure encryption scheme with message space $\mathcal{M}$ & key space K, then $|\mathcal{M}| \leq |K|$.

Proof: Assume $|K| < |\mathcal{M}|$.

Pick an arbitrary $c \in C$ where $Pr[C=c] > 0$.

$\mathcal{M}(c) := \{m \mid m = Dec_k(c) \text{ for some } k \in K\}$.

$|\mathcal{M}(c)| \leq |K| < |\mathcal{M}|$.

$\exists m' \in \mathcal{M}$ s.t. $m' \notin \mathcal{M}(c)$.

$Pr[M=m' \mid C=c] = 0 \neq Pr[M=m']$.

↑ possible for some distribution over $\mathcal{M}$