Chapter 1
Linear Programming

Paragraph 5
Duality

What we did so far

- We developed the 2-Phase Simplex Algorithm:
  - Hop (reasonably) from basic solution (bs) to bs until you find a basic feasible solution (bfs) or until it becomes clear that none exists.
  - Hop from bfs to bfs until you cannot find an improving solution or the improving trajectory is unbounded.
- We have seen that we can guarantee termination but that the worst-case runtime is exponential.
- Again: Note that LP is polynomial time solvable!

Lower Bounds on the Objective

- Let $\pi \in \mathbb{R}^m$, $\pi \geq 0$. Consider two sets:
  - $P_1 = \{x \in \mathbb{R}^n \mid Ax \geq b, x \geq 0\}$ and
  - $P_2 = \{x \in \mathbb{R}^n \mid \pi^T Ax \geq \pi^T b, x \geq 0\}$.
- Since $\pi \geq 0$, it is clear that $P_1 \subseteq P_2$, in other words: For all $x \in P_1$ we have that $\pi^T Ax \geq \pi^T b$.
- Now assume that $\pi \geq 0$ and also $\pi^T A \leq c^T$. Then, for all $x \in P_1$, we have that $\pi^T b \leq \pi^T Ax \leq c^T x$.
- Therefore, every such π provides us with a lower bound on the objective!

Duality

- The problem of finding the best such lower bound $\pi^T b$ with $\pi \in \hat{\mathrm{A}}$ such that $\pi^T A \leq c^T$, $\pi \geq 0$ is called the Dual Problem of (P_1, c).
- Formally, let $P_3 = \{\pi \in \hat{\mathrm{A}}^m \mid A^T \pi \leq c, \pi \geq 0\}$. Then, (P_3, b) is called the dual of the primal (P_1, c).
- Theorem
  - The dual of the dual is the primal.
  - For all primal feasible x and dual feasible π it holds that $\pi^T b \leq c^T x$ [weak duality].

Example

- Minimize $x_1+3x_2+x_3$ such that
  - $2x_1+x_2 \geq 3$
  - $-x_1+x_3 \geq 4$ (P)
  - $x_1, x_2, x_3 \geq 0$
- $\pi^{1^T} = (1/2,1) \Rightarrow$ Relax P_1 to
  - $0.5x_2+x_3 \geq 5.5$
  - $x_1, x_2, x_3 \geq 0$
- Maximize $3\pi_1+4\pi_2$ such that
  - $2\pi_1 - \pi_2 \leq 1$
  - $0 \leq \pi_1 \leq 3$ (D)
  - $0 \leq \pi_2 \leq 1$

π^1 is dual feasible and shows: every feasible x (if any exists!) has an objective value of at least 5.5!

Example

- Minimize $x_1+3x_2+x_3$ such that
  - $2x_1+x_2 \geq 3$
  - $-x_1+x_3 \geq 4$ (P)
  - $x_1, x_2, x_3 \geq 0$
- $\pi^{0T} = (1,1) \Rightarrow$ Relax P_1 to
  - $x1+x2+x3 \geq 7$
  - $x_1, x_2, x_3 \geq 0$
- Maximize $3\pi_1+4\pi_2$ such that
  - $2\pi_1 - \pi_2 \leq 1$
  - $0 \leq \pi_1 \leq 3$ (D)
  - $0 \leq \pi_2 \leq 1$

π^0 is dual optimal and shows: every feasible x (if any exists!) has an objective value of at least 7!

The Dual of the Standard Form

- $P_1 = \{x \in \hat{A}^n \mid Ax = b, x \geq 0\}$
  $(LP)_S = (P_1, c)$

- $P_2 = \{x \in \hat{A}^n \mid Ax \geq b, -Ax \geq -b, x \geq 0\}$
  $(LP)_C = (P_2, c)$

- $P_3 = \{(\eta, \mu) \in \hat{A}^m \mid A^T\eta - A^T\mu \leq c, \eta, \mu \geq 0\}$ C
  $P_4 = \{\pi \in \hat{A}^m \mid A^T\pi \leq c\}$ [π not non-negative anymore!]

- The dual of $(LP)_S$ is therefore (P_4, b).

The Dual of Unspecified Forms

- Min $c^T x$
  - $A_{M+}\, x \geq b_+$
  - $A_{M0}\, x = b_{\Leftrightarrow}$
  - $A_{M-}\, x \leq b_-$
  - $x_{N+} \geq 0$
  - x_{N0} unrestricted
  - $x_{N-} \leq 0$

$\longleftrightarrow$

- Max $b^T \pi$
  - $\pi_{M+} \geq 0$
  - π_{M0} unrestricted
  - $\pi_{M-} \leq 0$
  - $(A^T)_{N+}\, \pi \leq c_{N+}$
  - $(A^T)_{N0}\, \pi = c_{N0}$
  - $(A^T)_{N-}\, \pi \geq c_{N-}$

$M+ \cup M0 \cup M- = \{1,..,m\}$ $\quad$ $N+ \cup N0 \cup N- = \{1,..,n\}$

Strong Duality

$\bar{c}^T = c^T - c_B^T A_B^{-1} A$	$-c^T x^0$
$A_B^{-1} A$	$A_B^{-1} b$

- Given is an LP in standard form.
- We know: The current bfs x^0 in the tableau is optimal when $c^T - c_B^T A_B^{-1} A \geq 0$ … $(c_B^T A_B^{-1})\, A \leq c$.
- Therefore, $\pi^T := c_B^T A_B^{-1}$ is dual feasible!
- But: $\pi^T b = c_B^T A_B^{-1} b = c_B^T x_B^0 = c^T x^0$! ☺

Duality

- The problem of finding the best such lower bound $\pi^T b$ with $\pi \in \hat{A}$ such that $\pi^T A \leq c^T$, $\pi \geq 0$ is called the Dual Problem of (P_1,c).
- Formally, let $P_3 = \{\pi \in \hat{A}^m \mid A^T\pi \leq c, \pi \geq 0\}$. Then, (P_3,b) is called the dual of the primal (P_1,c).
- Theorem
  – The dual of the dual is the primal.
  – For all primal feasible x and dual feasible π it holds that $\pi^T b \leq c^T x$ [weak duality].
  – For primal optimal x^0 and dual optimal π^0 it holds that $\pi^{0^T} b = c^{0^T} x$ [strong duality].

Complementary Slackness

- Consider an LP in canonical form with x^0 primal optimal, and π^0 dual optimal.
- $c^T x^0 = {\pi^0}^T b \le {\pi^0}^T A x^0 \le c^T x^0 \;\Rightarrow\; {\pi^0}^T (Ax^0 - b) = 0.$
- ${\pi^0}^T b = c^T x^0 \ge {\pi^0}^T A x^0 \ge {\pi^0}^T b \;\Rightarrow\; (c^T - {\pi^0}^T A)\, x^0 = 0.$
- Consequently, since π^0, $Ax^0 - b \ge 0$, π^0 can only have non-zero components where there is no slack in the corresponding primal constraint.
- Analogously, since x^0, $c^T - {\pi^0}^T A \ge 0$, x^0 can only have non-zero components where there is no slack in the corresponding dual constraint.

Relation of Primal and Dual

Primal \ Dual	Finite Optimum	Unbounded	Infeasible
Finite Optimum	✓ [sd]	✗ [wd]	✗ [sd]
Unbounded	✗ [wd]	✗ [wd]	✓ [wd]
Infeasible	✗ [sd]	✓ [wd]	✓

wd – weak duality   sd – strong duality

Farkas' Lemma

- Given are $a_i \in$ Â^n, $1 \le i \le m$, and a vector $c \in$ Â^n.
- The cone of the a_i is defined as the set $C(a_1,..,a_m) := \{x \in$ Â$^n \mid x = \Sigma\, \pi_i a_i,\ \pi_i \ge 0\}$.
- Farkas' Lemma
  - (For all y with $y^T a_i \ge 0$ for all i it holds $y^T c \ge 0$) $\Leftrightarrow$
  - $c \in C(a_1,..,a_m)$.
- Proof: Exercise!

The Dual Simplex Algorithm

- We have seen how we compute a dual optimal solution after having solved the primal with the simplex algorithm.
- Consequently, we can solve the primal by solving the dual!
- In case that $c \geq 0$, $\pi = 0$ is dual feasible – so we may save the effort for solving phase I when focusing on the dual right away.
- Example: Transportation Problem, Diet Problem, Shortest Path, …

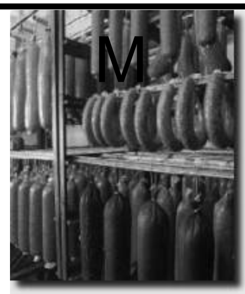
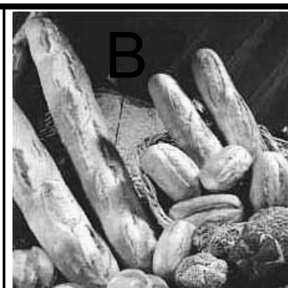
The Dual Simplex Algorithm

- Recall that the dual works by looking at $-\bar{A}^T$ instead of $\bar{A}$ in the tableau.
- Consequently, we can simply exchange the roles of rows and columns and consider matrix entries with opposite sign.
- Assume we are dual feasible. Then:
  - Search for a row with negative $\bar{b}_i$.
    (no such i $\Rightarrow$ dual optimal $\Rightarrow$ primal feasible)
  - Find a column j such that
    $\bar{a}_{ij} < 0$ and $\bar{c}_j$ / $-\bar{a}_{ij}$ is minimal!
    (no such j $\Rightarrow$ dual unbounded $\Rightarrow$ primal infeasible)
  - Pivot around $\bar{a}_{ij}$.

The Dual Simplex Algorithm

- Consider our instance of the diet problem:

Examples – Diet Problem

			Amount needed [g]
Cost/kg [$]	20	15	
Carbs [%]	10	25	500
Fat [%]	10	5	250
Protein [%]	15	20	600

Minimize 25M + 15B

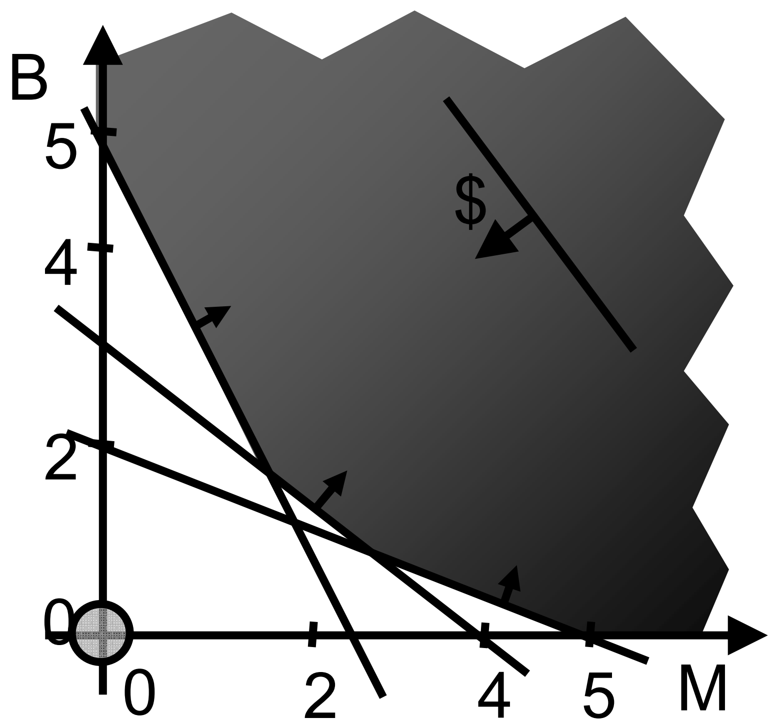
10M + 25B – 50

10M + 5B – 25

15M + 20B – 60

The Dual Simplex Algorithm

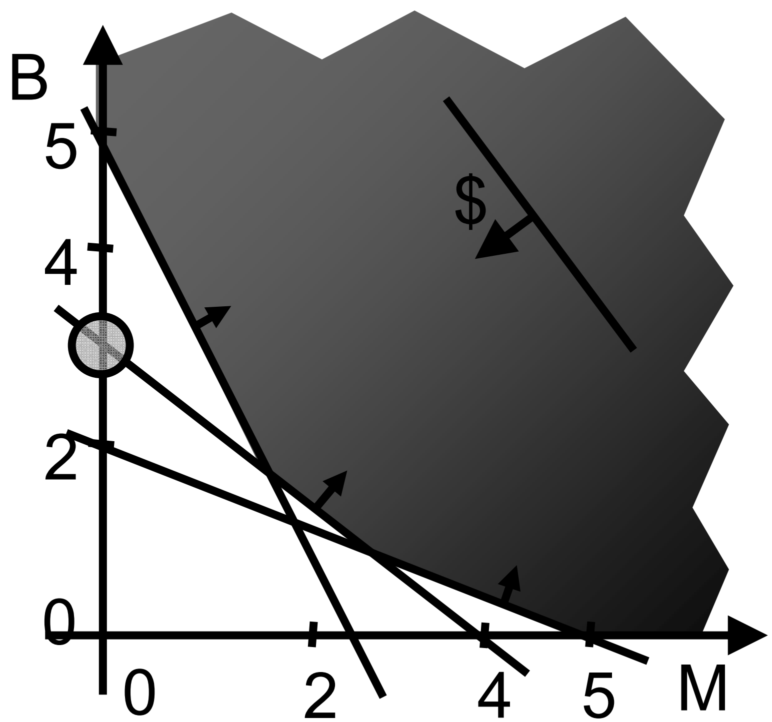
- Consider our instance of the diet problem:



0	0	0	25	15	0
1	0	0	-10	-25	-50
0	1	0	-10	-5	-25
0	0	1	-15	-20	-60

The Dual Simplex Algorithm

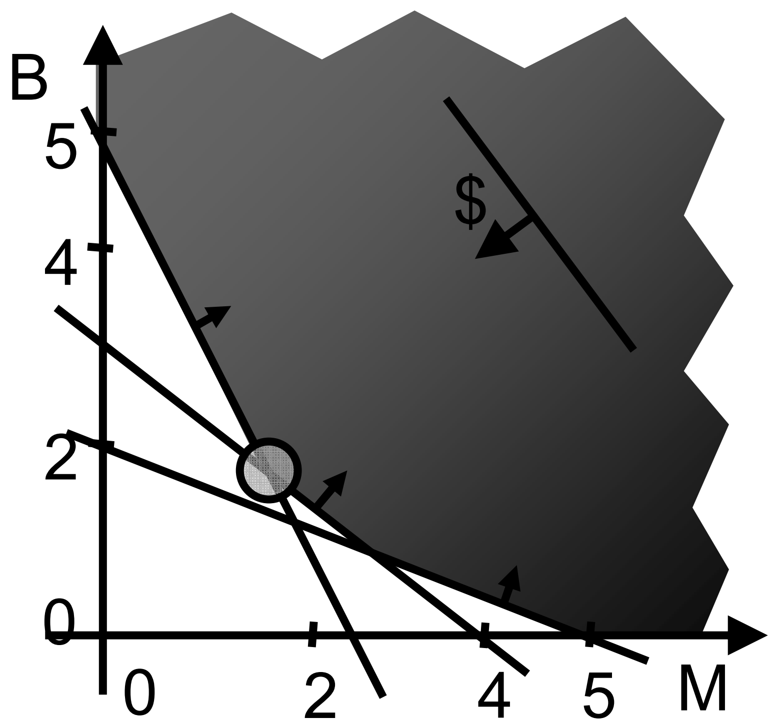
- Consider our instance of the diet problem:



0	0	3/4	55/4	0	-45
1	0	-5/4	35/4	0	25
0	1	-1/4	-25/4	0	-10
0		-1/20	3/4	1	3

The Dual Simplex Algorithm

- Consider our instance of the diet problem:



0	11/5	1/5	0	0	-67
1	7/5	-8/5	0	0	11
0	-4/25	1/25	1	0	8/5
0	3/25	-4/25	0	1	9/5

Shadow Prices

- Consider production planning. In the optimal tableau, the relative costs of the slack variables are exactly $-c_B^T A_B^{-1}$.
- Note that the objective in the simplex was to minimize. Thus, the relative costs of slack variables give us the dual solution.
- If a slack variable is not in the basis, it is 0 in the optimal solution. I.e. the corresponding constraint is tight for optimality.
- Now, the relative costs tell us, how much we could gain if the constraint was not tight. Consequently, the optimal duals are called “shadow prices” of their corresponding constraints.

Thank you!