Chapter 1
Linear Programming

Paragraph 2
Developing Ideas for an Algorithm solving Linear Optimization

What we did so far

- How to model an optimization problem
  - Data, Variables, Constraints, Objective
- We have seen that optimization consists of two tasks:
  - Finding a feasible solution and
  - Optimization
- Examples
  - Transportation Problem:
    specialized algorithm | optimality?
  - Diet Problem, Production Planning:
    geometrical solution | more than two/three variables?

What we did so far

- We established a formal definition what kind of problems we want to be able to solve: Linear Optimization Problems. A whole variety of optimization problems can be modeled that way!
- We established a standard form in which linear optimization problems can be modeled!

The Linear Optimization Problem

- Definition
  - Assume we are given a matrix $A \in \mathbb{R}^{m\times n}$, and vectors $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.
  - Setting $P_{A,b} := \{ x \in \mathbb{R}^n \mid Ax = b \text{ and } x \geq 0\}$ and $z_c(x) := c^Tx$, $(P_{A,b}, z_c)$ defines an optimization instance. Such an instance is called an instance of the Linear Optimization Problem. We say that it has Standard Form.
  - When setting $K_{A,b} := \{ x \in \mathbb{R}^n \mid Ax \geq b \text{ and } x \geq 0\}$ and $z_c(x) := c^Tx$, $(K_{A,b}, z_c)$ defines an optimization instance. Such an instance is also called an instance of the Linear Optimization Problem. We say that it has Canonical Form.

The Standard Form of Linear Optimization

- Lemma
  - Standard and canonical form of linear optimization are "equivalent".
  - Optimization problems for which not all components of the solution are required to be non-negative can be expressed in standard (or canonical) form.

Review

- Transportation Problem
  - We constructed a feasible solution first.
  - We then changed our solution while maintaining feasibility until no further improvement by re-routing was possible.
- Diet Problem and Production Planning
  - We modeled our problem with inequalities.
  - We visualized the feasible region graphically.
  - We solved our problem by improving the objective until any further improvement of the objective would have yielded to an infeasible solution.

Current Limitations

- The first algorithm sketch appears more general, but so far it only works for the transportation problem and optimality has not been proven.
- We do not know how to solve problems graphically when there are more than two variables involved.

Could we combine the geometrical view with ideas from our first method to achieve a more generally applicable solution method?

Towards a combination

- Let us try to develop a specialized algorithm production planning following the first idea.
- We can then apply the specialized algorithm and observe what happens in the geometrical interpretation.

Review

- Production Planning
  - given a set of resources that are available in limited amounts
  - given goods that we want to produce whereby for each good we need a specific amount of each resource
  - Every unit of each good produced yields a certain profit.
  - Decide what amount of each good should be produced such that the resources suffice and the profit is maximized!

Review

- Constants
  - a_r: maximum available amount of resource r
  - c_{rg}: how many units of resource r are needed for the production of one unit of good g
  - p_g: profit per unit of good g produced
- Variables
  - X_g: How many units of good g shall be produced?
- Constraints
  - $\Sigma_g X_g c_{rg} \leq a_r$ for all resources r
- Objective
  - Maximize $\Sigma_g p_g X_g$

Review

$$X_1 + 2X_2 \le 80$$

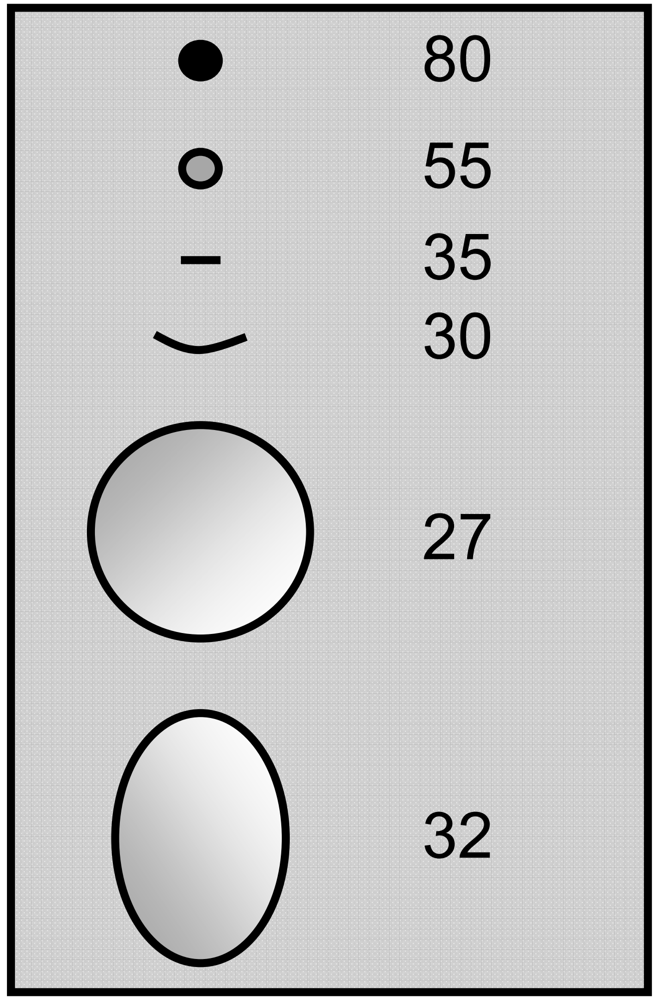
$$X_1 + X_2 \le 55$$

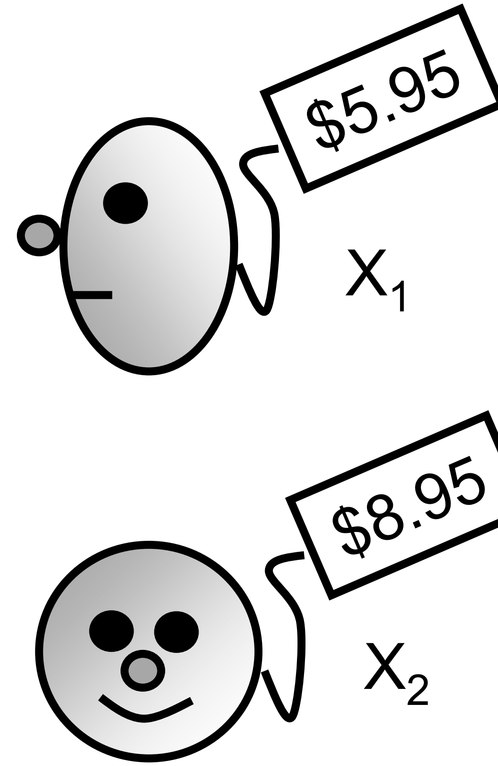
$$X_1 \le 35$$

$$X_2 \le 30$$

$$X_2 \le 27$$

$$X_1 \le 32$$





A Specialized Algorithm

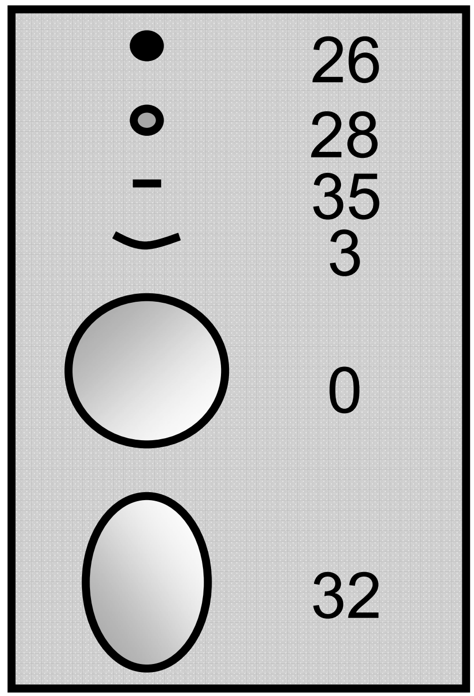
1. Construct a feasible solution first.
2. Change the solution while maintaining feasibility until no further improvement is possible.

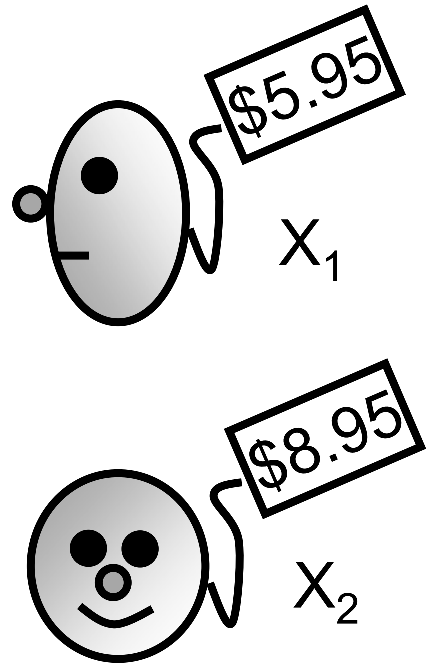
1a No production at all is feasible.

1b Produce as much of the product that yields the highest profit until the most limited resource is exhausted.

A Specialized Algorithm

$$X_1 + 2\,X_2 \le 80$$
$$X_1 + X_2 \le 55$$
$$X_1 \le 35$$
$$X_2 \le 30$$
$$X_2 \le 27$$
$$X_1 \le 32$$





X1 = 0	X1 = 0
X2 = 0	X2 = 27

A Specialized Algorithm

$X_1 + 2\,X_2 \leq 80$

$X_1 + X_2 \leq 55$

$X_1 \leq 35$

$X_2 \leq 30$

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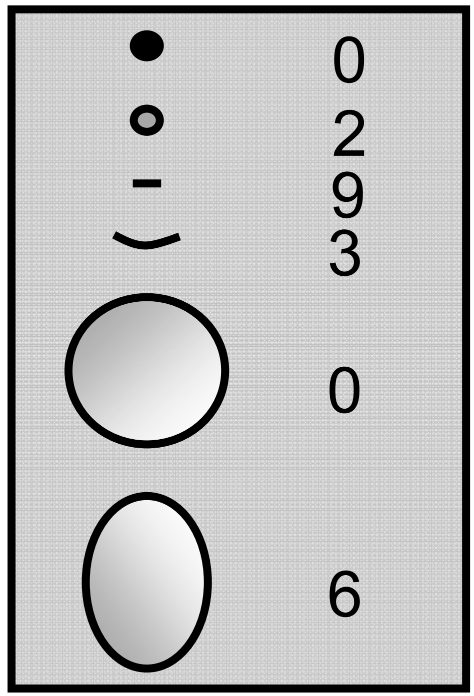

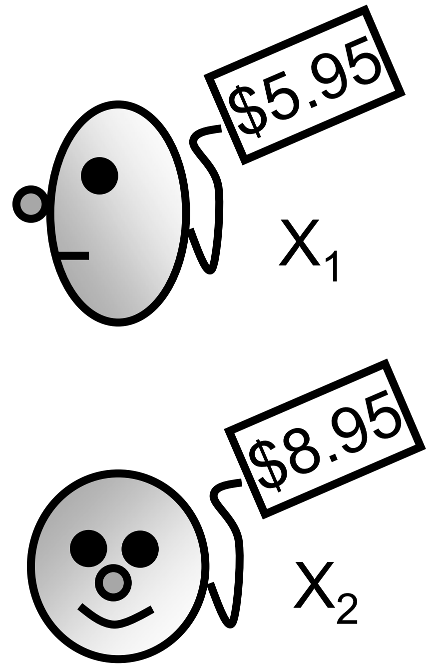

X1 = 0	X1 = 0	X1 = 26
X2 = 0	X2 = 27	X2 = 27

A Specialized Algorithm

$$X_1 + 2X_2 \le 80$$
$$X_1 + X_2 \le 55$$
$$X_1 \le 35$$
$$X_2 \le 30$$
$$X_2 \le 27$$
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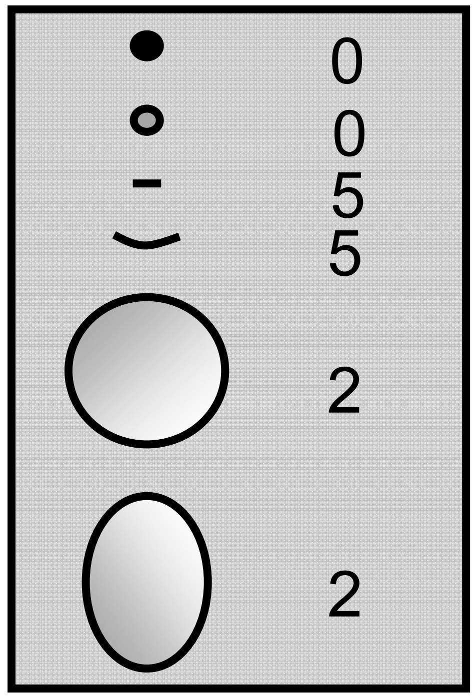
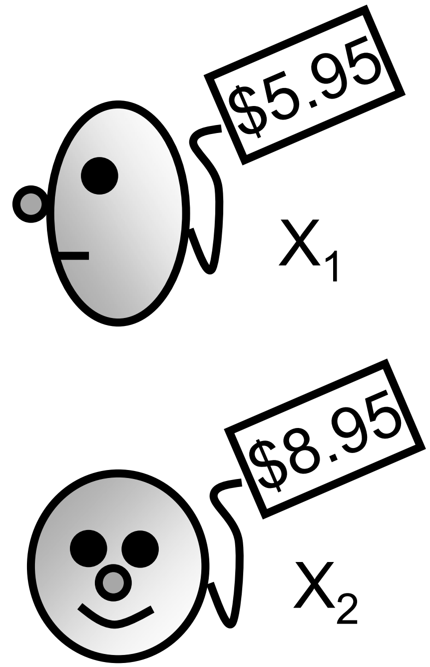




X1 = 0	X1 = 0	X1 = 26	X1 = 30
X2 = 0	X2 = 27	X2 = 27	X2 = 25

A Specialized Algorithm

1. Construct a feasible solution first.
2. Change the solution while maintaining feasibility until no further improvement is possible.

1a No production at all is feasible.

1b Produce as much of the product that yields the highest profit until the most limited resource is exhausted.

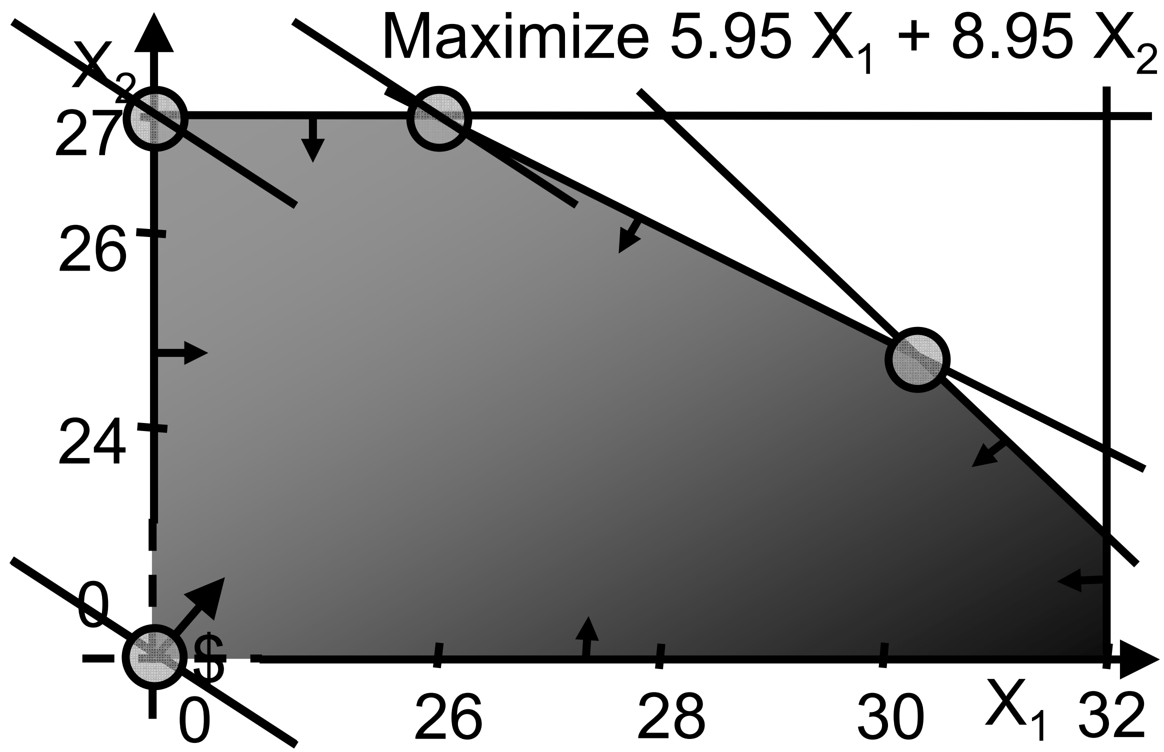
2 If it yields an increase, trade the more expensive product for the less expensive one while maintaining feasibility. Repeat.

What happens in the geometrical interpretation?

$X_1 + 2\,X_2 \leq 80$
$X_1 + \quad X_2 \leq 55$
$X_1 \qquad\quad \leq 35$
$\qquad\quad X_2 \leq 30$

$\qquad\quad X_2 \leq 27$

$X_1 \qquad\quad \leq 32$



X1 = 0	X1 = 0	X1 = 26	X1 = 30
X2 = 0	X2 = 27	X2 = 27	X2 = 25

Ananlysis

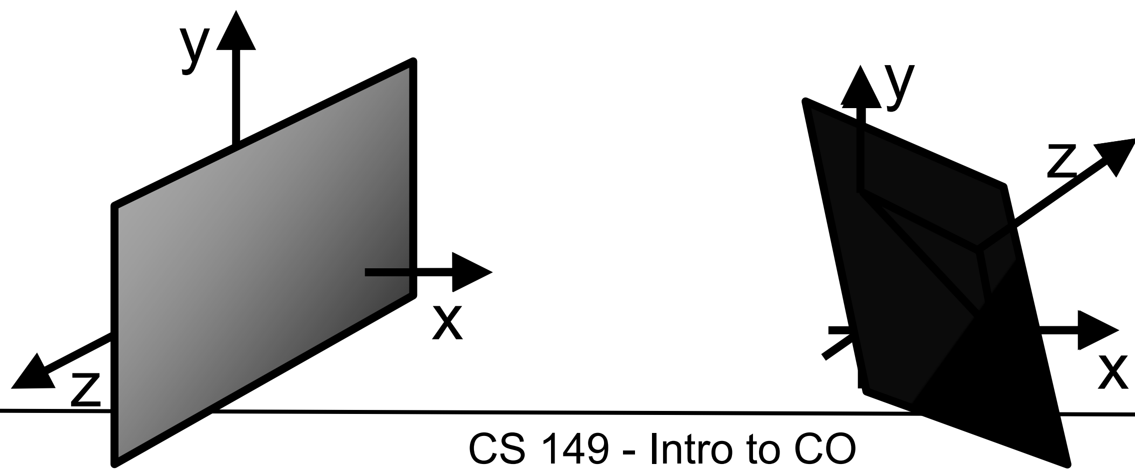
- Algorithm Idea

> Find a feasible corner (somehow).
>
> Check neighboring corners and see if one is better.
>
> Move over to the next corner until no better neighboring solution exists.

- Open Questions
  - Can we find a mathematical formalization of linear optimization problems for which we can define what "corner" and "neighboring corner" means?
  - Can we prove optimality?
  - How do we find a feasible starting solution?

What is a corner?

- A corner in our geometrical view is defined by the intersection of two lines. And a line is defined by an equation ⇨ a corner is a solution to an equation-system!
- What if there are more than two variables? How does an inequality look like then?
  - Given x,y,z, how does $x \geq 1$ look like?
  - Given x,y,z, how does $x+y+z = 1$ look like?



Equation Systems

- So an equation in an n-dimensional space defines an n-1-dimensional hyperplane!
  - n = 2: equations define lines
  - n = 3: equations define planes
- Every inequality divides the space in two halfspaces!
- A corner in an n-dimensional space is defined by the intersection of n hyperplanes. Therefore, a corner defines a solution to an equation system and vice versa.

Short Linear Algebra Review

- How to solve a linear equation system.
- Provide one solution – provide all solutions

Thank you!