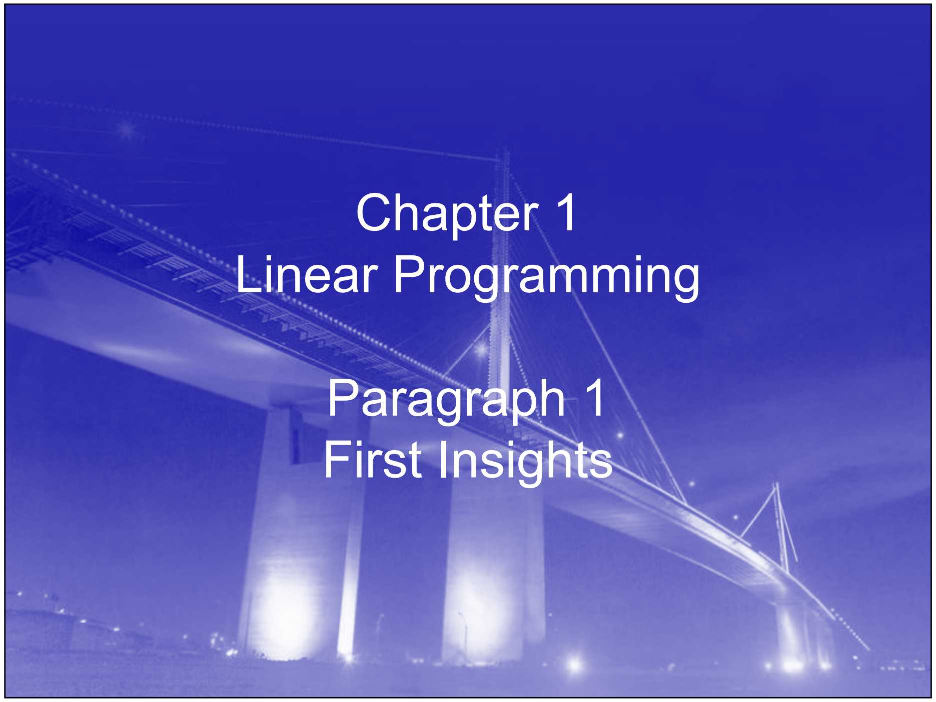


Chapter 1
Linear Programming

Paragraph 1
First Insights

What is Combinatorial Optimization?

Given a set of variables, each associated with a value domain, and given constraints over the variables, find an assignment of values to variables such that the constraints are satisfied and an objective function over the variables is minimized (or maximized)!

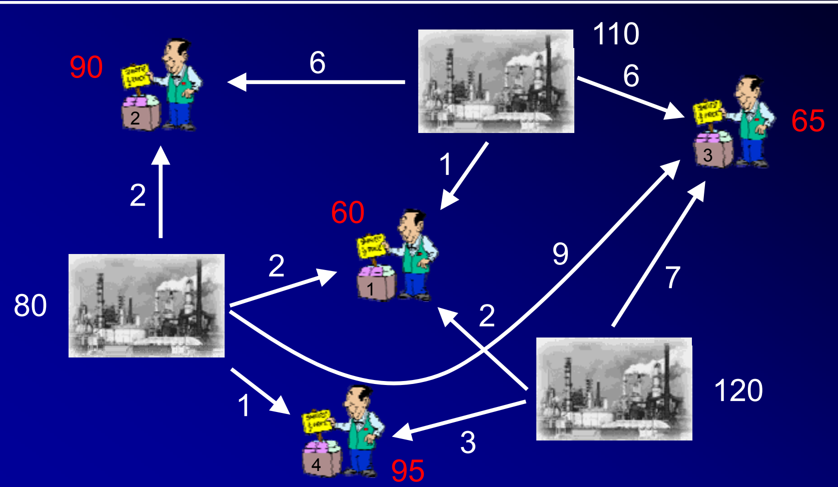
Examples

- Knapsack Problem
- Market Split Problem
- Network Problems
  - Maximum Flow
  - Minimum Spanning Tree
- Routing Problems
  - Shortest Path
  - Vehicle Routing
  - Travelling Salesman Problem
- Satisfiability Problem

Examples

- Transportation Problem: A good produced at various factories needs to be distributed to different retailers. Each factory provides a specific supply, and each retailer has a specific demand. Also, the transportation cost per unit of the good for each factory/retailer pair is known. How can the demand be met while minimizing transportation costs?

Examples – Transportation Problem

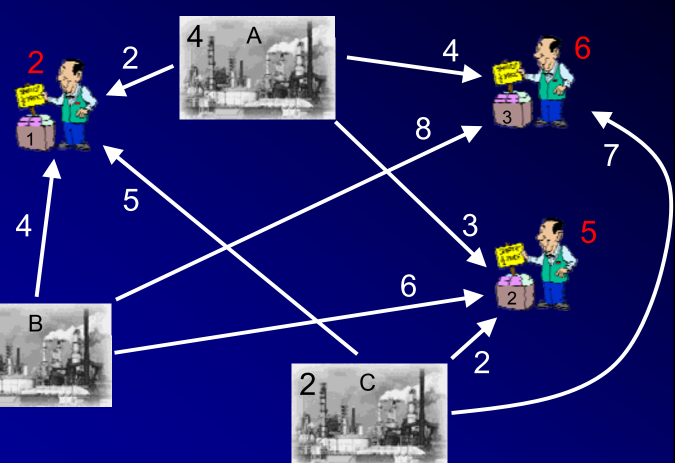


Examples – Transportation Problem

- Constants
  - D_r: demand of retailer r
  - S_f: supply of factory f
  - c_{fr}: cost of shipping one unit from f to r
- Variables
  - X_{fr}: How many units are sent from factory f to retailer r
- Constraints
  - $\sum_f X_{fr} = D_r$ for all retailers r
  - $\sum_r X_{fr} \le S_f$ for all factories f
- Objective
  - Minimize $\sum_{fr} c_{fr} X_{fr}$

How do we solve such problems?

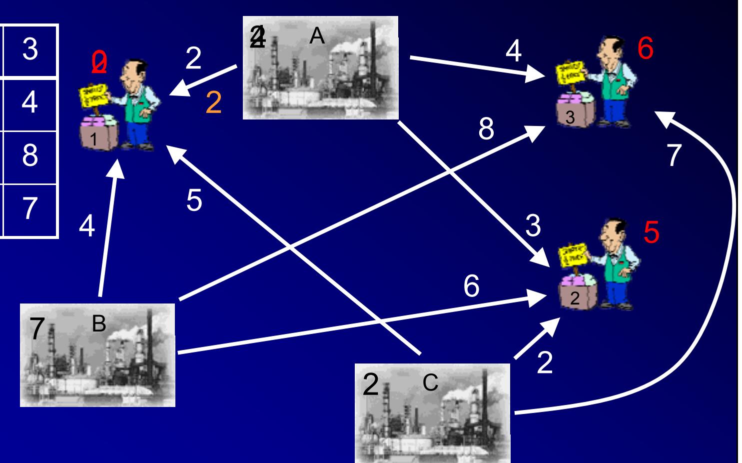
	1	2	3
A	2	3	4
B	4	6	8
C	5	2	7



Heuristic: Matrix-Minimum-Method

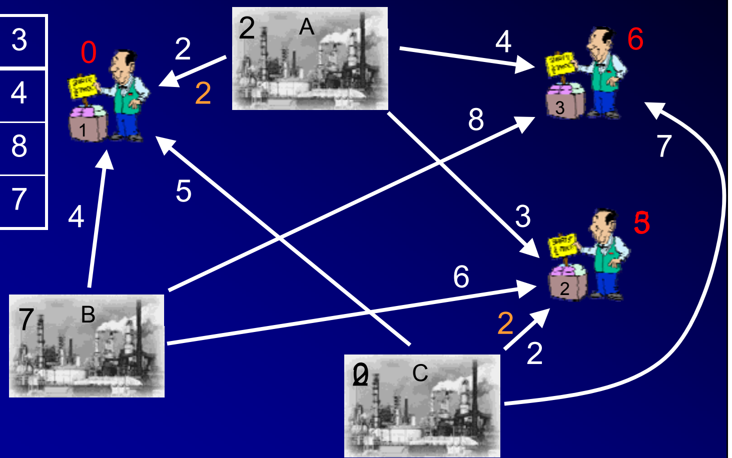
	1	2	3
A	2	3	4
B	4	6	8
C	5	2	7

Cost: 0



Heuristic: Matrix-Minimum-Method

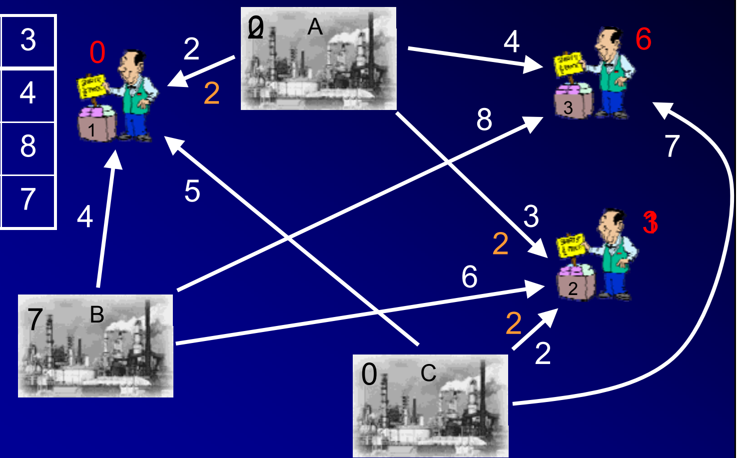
	1	2	3
A	2	3	4
B	4	6	8
C	5	2	7



Cost: 4

Heuristic: Matrix-Minimum-Method

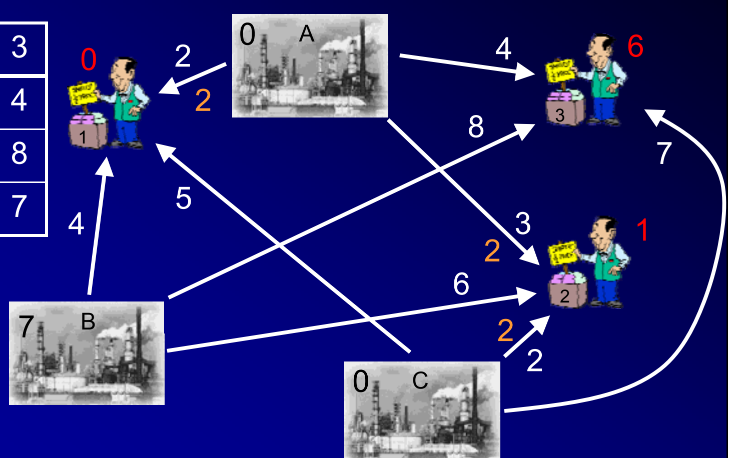
	1	2	3
A	2	3	4
B	4	6	8
C	5	2	7



Cost: 8

Heuristic: Matrix-Minimum-Method

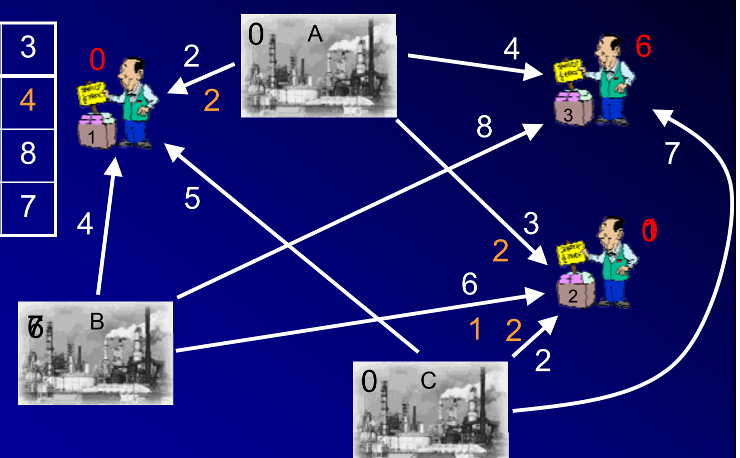
	1	2	3
A	2	3	4
B	4	6	8
C	5	2	7



Cost: 14

Heuristic: Matrix-Minimum-Method

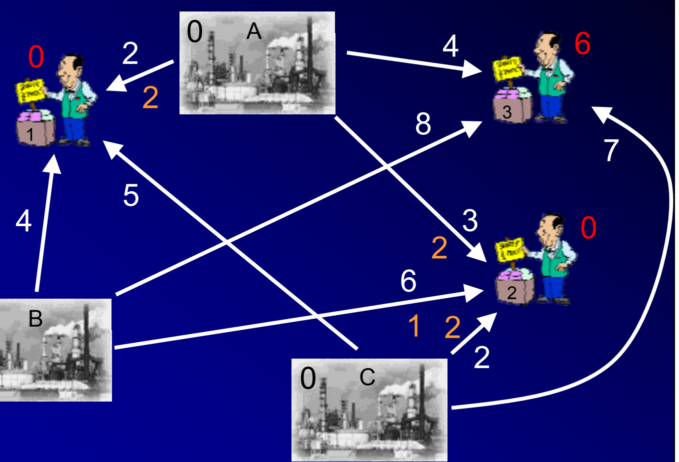
	1	2	3
A	2	3	4
B	4	6	8
C	5	2	7



Cost: 14

Heuristic: Matrix-Minimum-Method

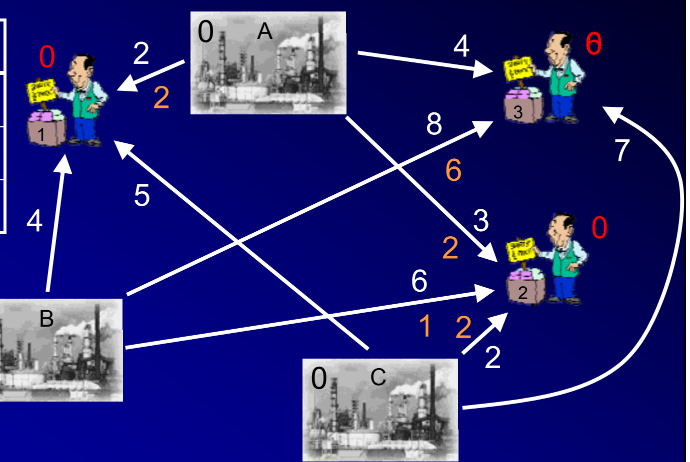
	1	2	3
A	2	3	4
B	4	6	8
C	5	2	7



Cost: 20

Heuristic: Matrix-Minimum-Method

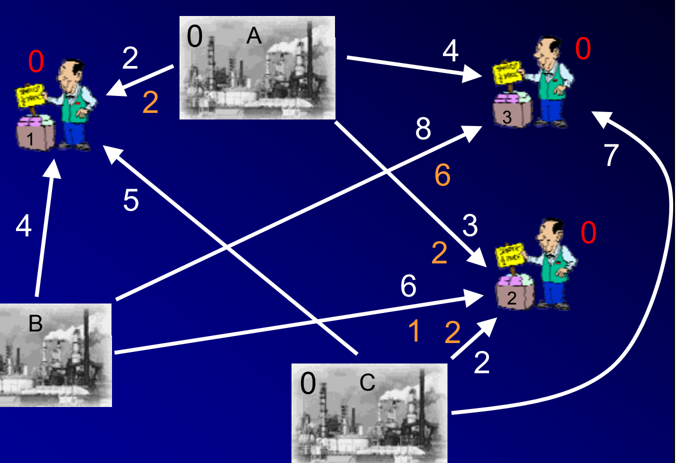
	1	2	3
A	2	3	4
B	4	6	8
C	5	2	7



Cost: 20

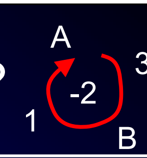
Heuristic: Matrix-Minimum-Method

	1	2	3
A	2	3	4
B	4	6	8
C	5	2	7

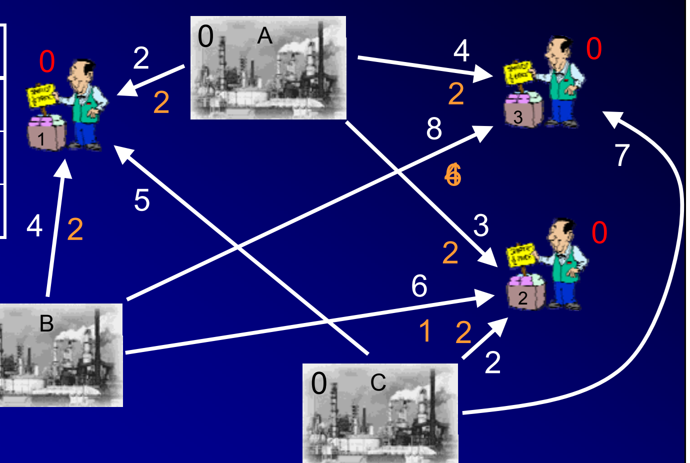


Cost: 68

Can we improve this solution?

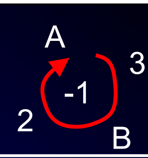


	1	2	3
A	2	3	4
B	4	6	8
C	5	2	7

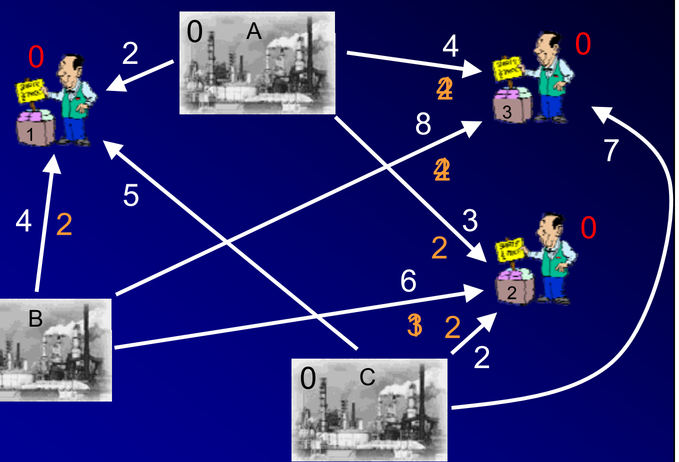


Cost: 68

Can we improve this solution?



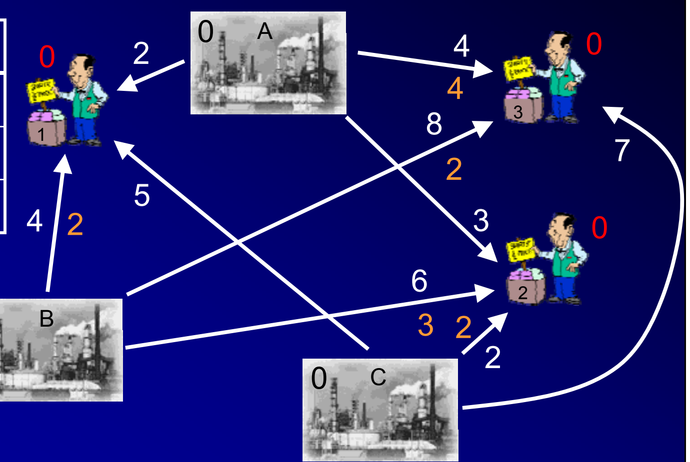
	1	2	3
A	2	3	4
B	4	6	8
C	5	2	7



Cost: 64

Can we improve this solution?

	1	2	3
A	2	3	4
B	4	6	8
C	5	2	7



Cost: 62

Analysis

- What did we do?
  - We constructed a feasible solution first.
  - We then changed our solution while maintaining feasibility until no further improvement by re-routing was possible.
- Open questions:
  - Is the solution that we found optimal?
  - If so, will our method always construct an overall optimal solution?
  - Can we generalize this procedure for other optimization problems?

Examples

- Diet Problem
  - given a set of foods
    - each containing a certain amount of ingredients (vitamins, calories, minerals, etc)
    - each food associated with a certain price per kg
  - Each ingredient needs to be provided in sufficient amounts in order to survive.
  - Decide what amounts of each food should be purchased so that survival is guaranteed while the costs are minimized!

Examples – Diet Problem

	 M	 B	Amount needed [g]
Cost/kg [$]	25	15	
Carbs [%]	10	25	500
Fat [%]	10	5	250
Protein [%]	15	20	600

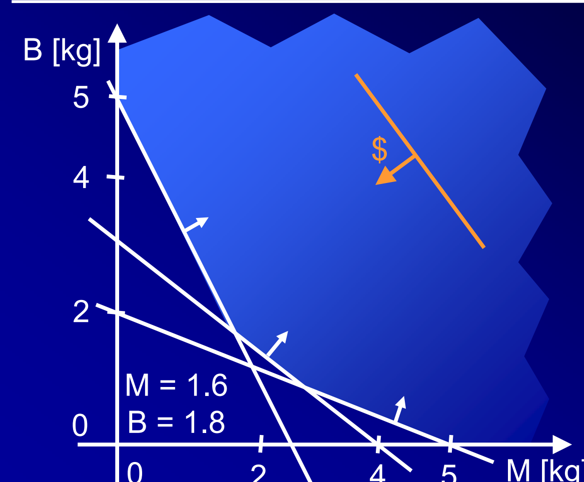
Examples – Diet Problem

- Constants
  - m_i: critical minimum amounts that are needed for each ingredient i
  - a_{if} : amount of each ingredient i in food f
  - c_f: cost per kg of food f
- Variables
  - X_f: How many kg of food f shall be purchased?
- Constraints
  - Î$_f$ $X_f a_{if}$ – m_i for all ingredients i
- Objective
  - Minimize Î$_f$ $c_f X_f$

Examples – Diet Problem

	 M	 B	Amount needed [g]	
Cost/kg [$]	25	15		Minimize 25M + 15B
Carbs [%]	10	25	500	100M + 250B – 500
Fat [%]	10	5	250	100M + 50B – 250
Protein [%]	15	20	600	150M + 200B – 600

Examples – Diet Problem



Minimize 25M + 15B

100M + 250B – 500

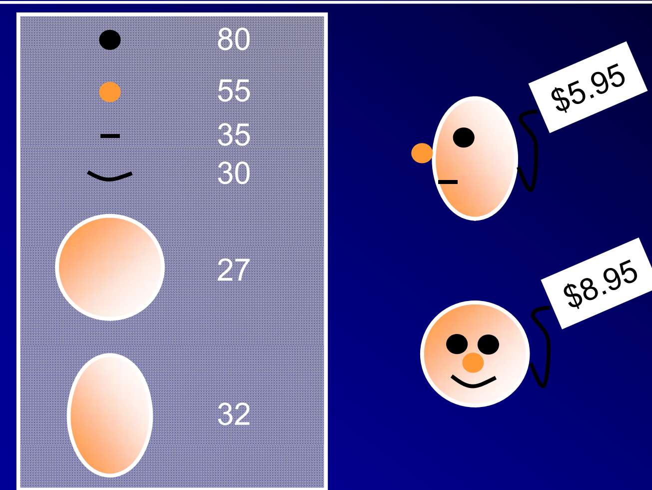
100M + 50B – 250

150M + 200B – 600

Examples

- Production Planning
  - given a set of resources that are available in limited amounts
  - given goods that we want to produce whereby for each good we need a specific amount of each resource
  - Every unit of each good produced yields a certain profit.
  - Decide what amount of each good should be produced such that the resources suffice and the profit is maximized!

Examples – Production Planning



Examples – Production Planning

- Constants
  - a_r: maximum available amount of resource r
  - c_{rg}: how many units of resource r are needed for the production of one unit of good g
  - p_g: profit per unit of good g produced
- Variables
  - X_g: How many units of good g shall be produced?
- Constraints
  - $\hat{\text{I}}_g X_g c_{rg} \leq a_r$ for all resources r
- Objective
  - Maximize $\hat{\text{I}}_g p_g X_g$

Examples – Production Planning

$X_1 + 2X_2 \leq 80$

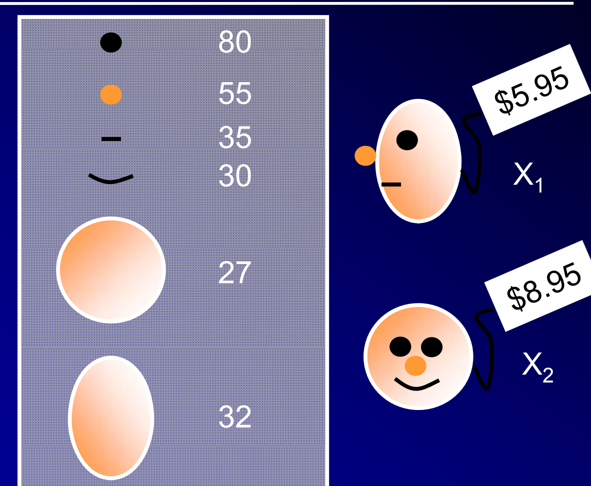
$X_1 + X_2 \leq 55$

$X_1 \leq 35$

$X_2 \leq 30$

$X_2 \leq 27$

$X_1 \leq 32$



Examples – Production Planning

$X_1 + 2\,X_2 \le 80$

$X_1 + \quad X_2 \le 55$

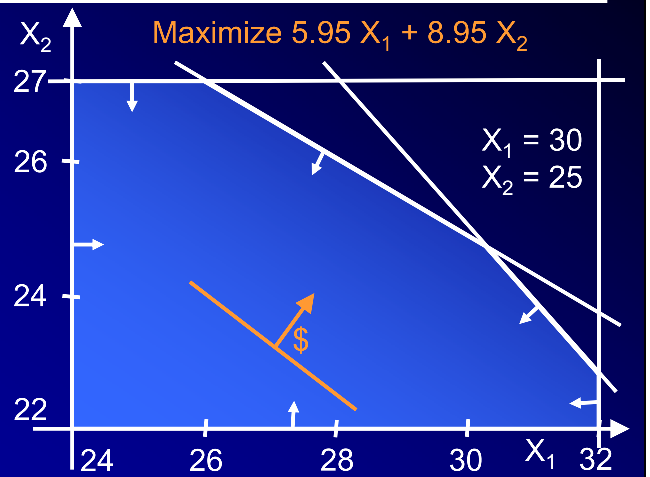
$X_1 \qquad \le 35$

$X_2 \le 30$

$X_2 \le 27$

$X_1 \qquad \le 32$

X_1, X_2 – 0



Analysis

- What did we do?
  - We modeled our problem with inequalities.
  - We visualized the feasible region graphically.
  - We solved our problem by improving the objective until any further improvement of the objective would have yielded to an infeasible solution.
- Open questions
  - How can we solve problems when there are more than two variables involved?
  - Could we combine the geometrical view with ideas from our first method to achieve a more generally applicable solution method?

Optimization Problems

- Definition
  - An Optimization Instance is a pair (P,z) whereby P is a set of n-tuples (the set of feasible solutions) and a cost-function z: P → Â. A solution x µ P is called optimal if and only if x = argmin z(P).
  - An Optimization Problem is a set of optimization instances.
  - An algorithm solves an optimization problem if and only if it computes an optimal solution for every instance of the optimization problem or shows that none exists.

Optimization Problems - Examples

- Given a weighted, undirected graph G, let us denote with S_G the set of all spanning trees in G. Further, denote with z_G the function that assigns the corresponding cost to a spanning tree.
- For all undirected graphs G, (S_G,z_G) is an optimization instance.
- The set { (S_G,z_G) | G weighted, undirected graph} is the optimization problem of finding a minimum spanning tree in a given graph.
- Prim's and Kruskal's algorithms solve this optimization problem.

The Linear Optimization Problem

- Definition
  – Assume we are given a matrix A µ $Â^{mxn}$, and vectors b µ $Â^{m}$, and c µ $Â^{n}$.
  – Setting $P_{A,b}$:= { x µ $Â^{n}$ | Ax = b and x – 0} and $z_c(x)$:= $c^T x$, ($P_{A,b}$,z_c) defines an optimization instance. Such an instance is called an instance of the Linear Optimization Problem. We say that it has Standard Form.
  – When setting $K_{A,b}$:= { x µ $Â^{n}$ | Ax – b and x – 0} and $z_c(x)$:= $c^T x$, ($K_{A,b}$,z_c) defines an optimization instance. Such an instance is also called an instance of the Linear Optimization Problem. We say that it has Canonical Form.

The Standard Form of Linear Optimization

- Lemma
  – Standard and canonical form of linear optimization are “equivalent”.
  – Optimization problems for which not all components of the solution are required to be non-negative can be expressed in standard (or canonical) form.

Thank you!