Chapter 1
Linear Programming

Paragraph 5
Duality

What we did so far

- We developed the 2-Phase Simplex Algorithm.
- We introduced the notion of Duality and established a close relationship between an LP and its dual:
  - Weak duality: Every dual feasible solution provides a bound on the objective.
  - Strong duality: Optimal primal and optimal dual solutions have the same objective value.
- We developed the Dual Simplex Algorithm.

Why Different Variants of Simplex?

- The dual simplex saves the hassle of finding a feasible solution first. It is especially well suited for re-optimization after new constraints are added.

Another Simplex-Variant

- Recall the complementary slackness property:
  - x^0 and π^0 are optimal $\Leftrightarrow$
  - ${\pi^0}^T(Ax^0-b) = 0$ and $(c^T - {\pi^0}^T A)\, x^0 = 0$.
- Idea: We could try to modify both primal and dual variable instantiations such that we come “closer” to fulfilling the complementary slackness conditions. If we succeed in meeting them, then we have solved the problem!

The Standard Form and its Dual

- Min $c^T x$
  - $Ax = b$ (P)
  - $x \geq 0$
- Max $b^T \pi$
  - π unrestricted (D)
  - $A^T \pi \leq c$

- Assume that $b \geq 0$ and that we know a feasible dual solution π.
- Denote with $J(\pi) := \{ j \mid A_j^T \pi = c_j \}$ the set of admissible columns.
- Now assume we found a feasible solution x for (P) such that $\{ j \mid x_j > 0 \} \subseteq J(\pi)$, then both π and x are optimal!

How to find such an x?

- Thus, we should try to find a solution to the following set of constraints:
  - $Ax = b$
  - $x_j = 0$ for all $j \notin J(\pi)$
  - $x \geq 0$
- Finding a feasible solution to such a system is something that we have done already in the 2-phase simplex algorithm:
  - Min $1^T x^a$
  - $A_J x_J + x^a = b$ (RP)
  - $x_J, x^a \geq 0$

Properties of the Restricted Primal (RP)

- Since $b \geq 0$, $x^a = b$ and $x_J = 0$ defines a feasible solution.
- Since $x^a \geq 0$, the objective is bounded from below.
- Consequently, (RP) has an optimal solution, and so has its dual, that looks as follows:
  - Max $b^T\pi$
  - $\pi \leq 1$ (DRP)
  - $A_J^T\pi \leq 0$.

Improving the Dual Solution

- If (RP) has an optimal objective > 0, then this proves that π cannot be optimal.
- Let π^r denote the optimal solution of (DRP).
- Theorem
  – There exists a $\lambda > 0$ such that $\pi^* := \pi + \lambda\pi^r$ is feasible for (D) with $b^T\pi^* > b^T\pi$.
- Proof:

The Primal-Dual Algorithm

- Determine an initial π that is feasible for (D).
- Determine $J(\pi)$ and the corresponding RP.
- Solve RP and compute its optimal dual solution π^r.
- If RP has objective 0, we are done and the corresponding $(x_J, 0)$ solves (P).
- Otherwise, compute λ maximal so that $\pi^* = \pi + \lambda \pi^r$ is dual feasible.
- Set $\pi := \pi^*$ and continue.

An Example

- Solve Min $2x_1+2x_2+3x_3$ such that
  - $x_1+x_3 \geq 1$
  - $x_2+x_3 \geq 2$
  - $x_{1,},x_2,x_3 \geq 0$
- Min $2x_1+2x_2+3x_3$ such that
  - $-s_1+x_1+x_3 = 1$
  - $-s_2+x_2+x_3 = 2$
  - $s_1,s_2,x_{1,},x_2,x_3 \geq 0$

An Example

- Min $2x_1+2x_2+3x_3$ such that
  - $-s_1+x_1+x_3 = 1$
  - $-s_2+x_2+x_3 = 2$
  - $s_1,s_2,x_{1,},x_2,x_3 \geq 0$

a_1	a_2	s_1	s_2	x_1	x_2	x_3	⊠
1	1	0	0	0	0	0	0
1		-1		1		1	1
	1		-1		1	1	2

An Example

- Min $2x_1+2x_2+3x_3$ such that
  - $-s_1+x_1+x_3 = 1$
  - $-s_2+x_2+x_3 = 2$
  - $s_1,s_2,x_1,x_2,x_3 \geq 0$
- $\pi^T = (0,0)$
- $c^T - \pi^T A = (0,0,2,2,3)$
- $J = \{s_1,s_2\}$
- $\pi^r = (1,1) – (0,0) = (1,1)$
- $\lambda = \min \{2/1,2/1,3/2\} = 3/2$

a_1	a_2	s_1	s_2	x_1	x_2	x_3	⊠
0	0	1	1	-1	-1	-2	-3
1		-1		1		1	1
	1		-1		1	1	2

An Example

- Min $2x_1+2x_2+3x_3$ such that
  - $-s_1+x_1+x_3 = 1$
  - $-s_2+x_2+x_3 = 2$
  - $s_1,s_2,x_{1,},x_2,x_3 \geq 0$
- $\pi^T = (3/2,3/2)$
- $c^T - \pi^T A = (3/2,3/2,1/2,1/2,0)$
- $J = \{x_3\}$
- $\pi^r = (1,1) – (2,0) = (-1,1)$
- $\lambda = \min \{3/2,1/2\} = 1/2$

a_1	a_2	s_1	s_2	x_1	x_2	x_3	⊠
0	0	1	1	-1	-1	-2	-3
1		-1		1		1	1
	1		-1		1	1	2

An Example

- Min $2x_1+2x_2+3x_3$ such that
  - $-s_1+x_1+x_3 = 1$
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  - $s_1,s_2,x_{1,},x_2,x_3 \geq 0$

- $\pi^T = (3/2,3/2)$
- $c^T - \pi^T A = (3/2,3/2,1/2,1/2,0)$
- $J = \{x_3\}$
- $\pi^r = (1,1) – (2,0) = (-1,1)$
- $\lambda = \min \{3/2,1/2\} = 1/2$

a_1	a_2	s_1	s_2	x_1	x_2	x_3	⊠
2	0	-1	1	1	-1	0	-1
1		-1		1		1	1
-1	1	1	-1	-1	1	0	1

An Example

- Min $2x_1+2x_2+3x_3$ such that
  - $-s_1+x_1+x_3 = 1$
  - $-s_2+x_2+x_3 = 2$
  - $s_1,s_2,x_{1,},x_2,x_3 \geq 0$
- $\pi^T = (1,2)$
- $c^T - \pi^T A = (1,2,1,0,0)$
- $J = \{x_{2,}x_3\}$
- $x^T = (0,1,1)$

a_1	a_2	s_1	s_2	x_1	x_2	x_3	⊠
2	0	-1	1	1	-1	0	-1
1		-1		1		1	1
-1	1	1	-1	-1	1	0	1

An Example

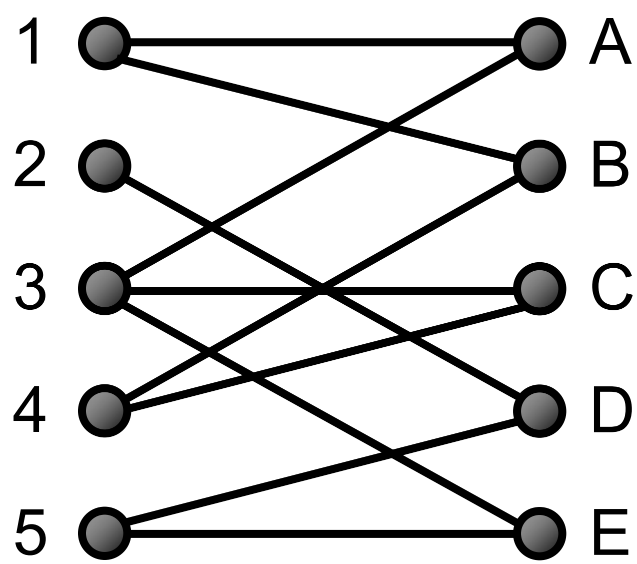
- Min $2x_1+2x_2+3x_3$ such that
  - $-s_1+x_1+x_3 = 1$
  - $-s_2+x_2+x_3 = 2$
  - $s_1,s_2,x_1,x_2,x_3 \geq 0$
- $\pi^T = (1,2)$
- $c^T - \pi^T A = (1,2,1,0,0)$
- $J = \{x_2, x_3\}$
- $x^T = (0,1,1)$

a_1	a_2	s_1	s_2	x_1	x_2	x_3	☒
1	1	0	0	0	0	0	0
1		-1		1		1	1
-1	1	1	-1	-1	1	0	1

Why Different Variants of Simplex?

- The dual simplex saves the hassle of finding a feasible solution first. It is especially well suited for re-optimization after new constraints are added.
- The primal-dual algorithm derives its importance from the simplicity of the restricted problem or its dual, respectively (much simpler objective or right-hand side!). Often it can be solved with a simple, specialized algorithm.

Min-Cost Perfect Matching



	A	B	C	D	E
1	1	6			
2		1		7	
3	2		9		2
4		3	8		
5				1	5

Min-Cost Perfect Matching

- Given a weighted bipartite graph $G=(V_1 \cup V_2,E,c)$.
- Min $\Sigma_{(i,j) \in E}\, c_{ij}\ \ x_{ij}$ such that
  - $\Sigma_{j;(i,j) \in E}\, x_{ij} = 1$ for all $i \in V_1$ | π
  - $\Sigma_{i;(i,j) \in E}\, x_{ij} = 1$ for all $j \in V_2$ | ❍
  - $x \geq 0$
- Given a dual feasible solution (π, ❍), we have F := J(π, ❍) = { $(i,j) \in E$ | π_i + ❍$_j$ = c_{ij} }. Then, RP(π, ❍) is:
- Min $\Sigma_i a_i\ + \Sigma_j b_j$ such that
  - $a_i + \Sigma_{j;(i,j) \in F}\, x_{ij} = 1$ for all $i \in V_1$
  - $b_j + \Sigma_{i;(i,j) \in F}\, x_{ij} = 1$ for all $j \in V_2$
  - $a,b,x \geq 0$

Min-Cost Perfect Matching

- Thus, RP(π, ❍) is a maximum matching problem!
- It can be shown that every successive matching contains at least one more edge.
- Consequently, the primal-dual algorithm reduced the problem of finding a min-cost perfect matching to solving a sequence of at most $|V_1|$ maximum bipartite matching problems.
- While maximum bipartite matching can be solved in $O(|V_1|^{1/2} E)$, a more careful analysis of the primal dual algorithm even shows that it solves the min-cost perfect matching problem in time $O(|V_1+V_2|^3)$.

Thank you!