Chapter 2.5

Intermission
Zero-Sum Games

Zero-Sum Games

- A game consists of
  - Players: Can be people, companies, states, or even “randomness”.
  - Moves: Players can make moves (in some order or at the same time) according to the rules of the game. Generally, a move is a selection from a set of actions.
  - Strategy: Is a vector that gives the probability of choosing an action at a given state. If one action has probablity 1, we speak of a pure strategy, otherwise, it's a mixed strategy.
  - Reward: The gain/loss of a player given a certain outcome of the game. In the case of 2-player games, the reward takes the form of a matrix.

Zero-Sum Games

- Definition
  - 2-person zero-sum game is a game where the gain of one player equals the loss of the other. We also speak of a matrix game in this case.
  - A matrix game is called symmetric if the set of actions for both players is the same AND the matrix is anti-symmetric ($a_{ij} = -a_{ij}$).
- Convention
  - The reward matrix reflects the view of player 1.

Zero-Sum Games

- Example - Schnick-Schnack-Schnuck (also boringly known as Rock, Paper, Scissors):

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Zero-Sum Games

- Example - Store Location:
  - WalMart and K-Mart can choose in which out of 4 cities in RI (Providence, Warwick, Newport, and Cranston) they should open a store.

	P	W	N	C
P	-6	22	14	14
W	-2	8	10	15
N	14	18	22	25
C	6	14	10	9

Zero-Sum Games

- Let $S^n = \{ x \in \mathbb{R}^n \mid x \geq 0, 1^T x = 1 \}$ the set of strategies
  - $x_j = P(P_1 = \text{action}_j)$ and $y_i = P(P_2 = \text{action}_i)$
- We assume that the the players act independently from another, i.e.:
  - $P(P_1 = \text{action}_j \text{ and } P_2 = \text{action}_i) = x_j * y_i$
- Then, the expected outcome of a game is
  - $\mathbb{E}(y,x) = \sum_{i,j} a_{ij} P(P_1 = \text{action}_j \text{ and } P_2 = \text{action}_i) = \sum_{i,j} y_i * a_{ij} * x_j = y^T A x.$

Zero-Sum Games

- Player 1 may search for a strategy such that his profit is maximized against a worst-case adversary:
  - Find x^0 such that $\mathbb{E}(y,x^0) = \max_x \min_y \mathbb{E}(y,x)$.
  - $M_0 := \max_x \min_y \mathbb{E}(y,x)$ is called the value of the game, and x^0 an optimal strategy for player 1.
  - We also define, from player 2's viewpoint: $M^0 := \min_y \max_x \mathbb{E}(y,x)$.

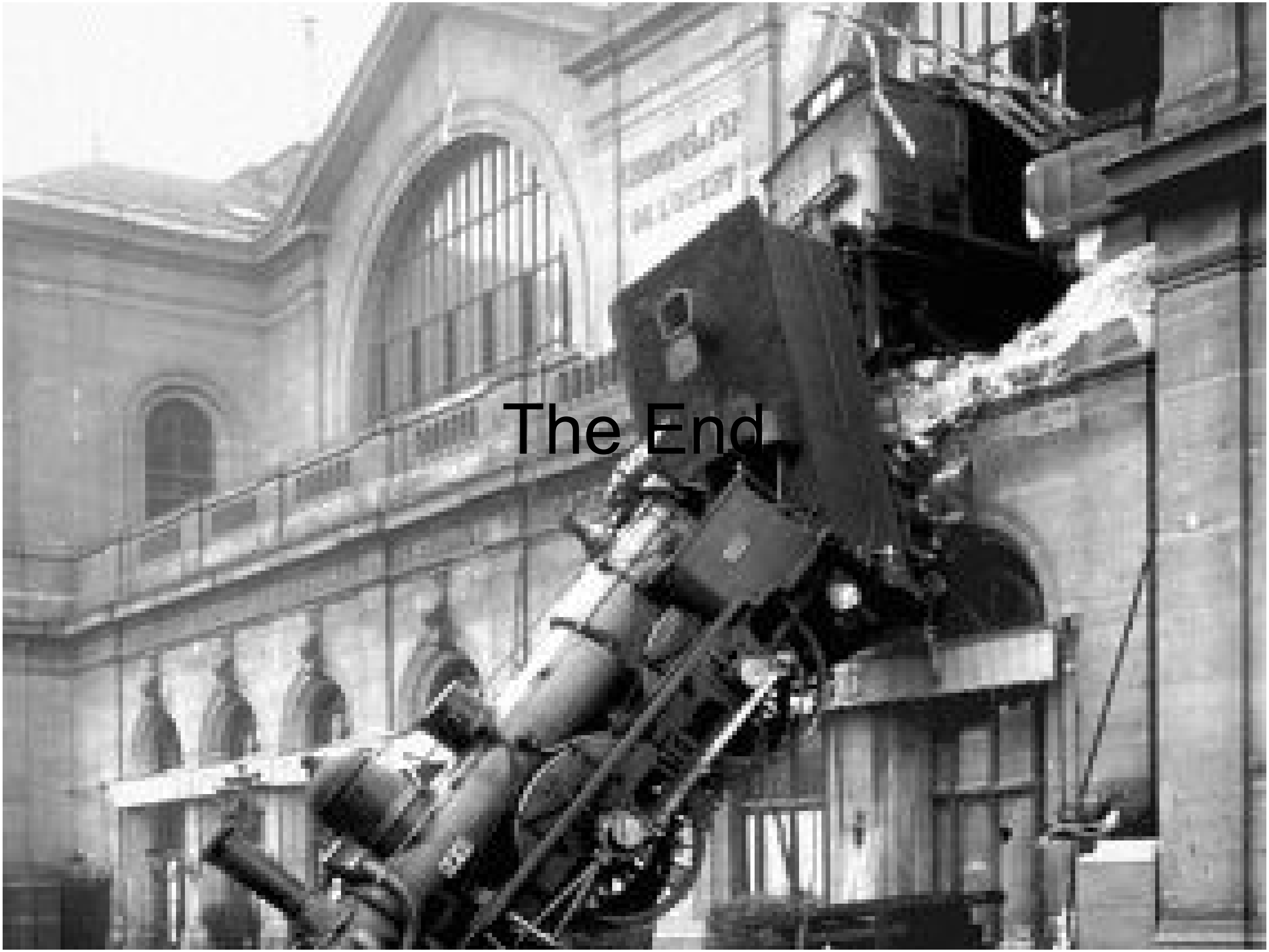
Zero-Sum Games

- Remark:
  - If we fix a strategy $x^1 \in S^n$ for player 1, what is the optimal response for player 2?
  - $\min_y y^T A x^1 = \min_y \sum_{i,j} y_i (A_i x^1) = \min_i (A_i x^1)$
  - Consequently, there always exists a pure strategy that is optimal as a response to a known strategy of the opponent!

Zero-Sum Games

- As a consequence, we can compute an optimal strategy for player 1 with the help of linear programming:
  - $\max_x \mathbb{E}(y,x) \quad = \quad \max z$

$$\text{such that } A_i x \geq z \qquad \text{for all } i$$
$$1^T x = 1$$
$$x \geq 0$$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