Chapter 2
Integer Programming

Paragraph 2
Branch and Bound

What we did so far

- We studied linear programming and saw that it is solvable in P.
- We gave a sufficient condition (total unimodularity) that simplex will return an integer solution.
  - Shortest Path
  - Minimum Spanning Tree
  - Maximum Flow
  - Min-Cost Flow
- How can we cope with general integer programs?

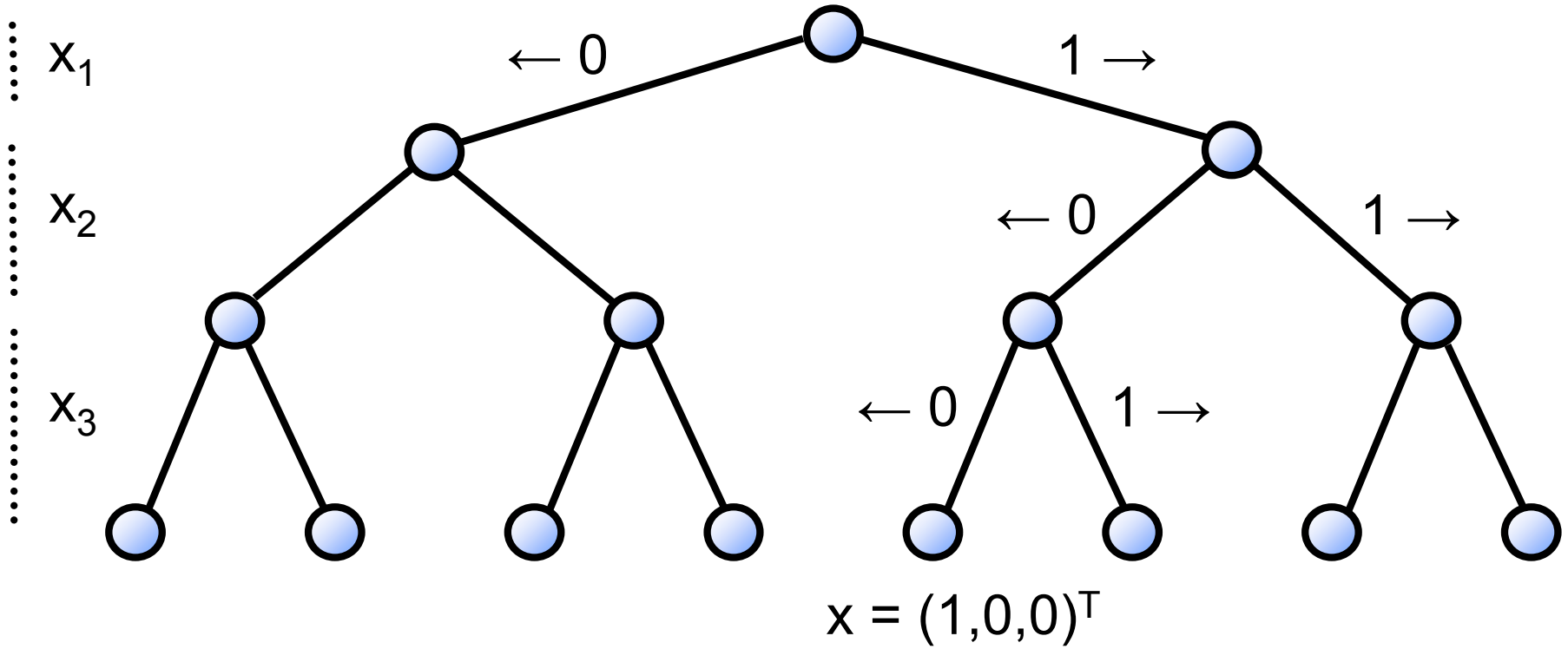
Tree Search

- As an example, assume we have to solve the Knapsack Problem.
- Recall that there are 2^n possible combinations of knapsack items.
- The brute-force approach to solve the problem is to enumerate all combinations, see which ones are feasible, and which one of those achieves maximum profit.
- A systematic way of enumerating all solutions is via backtracking.

Tree Search

- Assume we order the variables $x_1,..,x_n$.
- A recursive way of enumerating all solutions is to set x_1 to 0 first and to recursively enumerate all solutions for $KP(x_2,..,x_n, p, w, C)$. Then we set x_1 to 1 and enumerate all solutions for $KP(x_2,..,x_n, p, w, C-w_1)$.
- This procedure yields to a search tree!

Tree Search



Combinatorial Explosions

- Enumerating all possible solutions is of course not feasible when there are too many items.
- What is “too many”?
  - 500? 200? 100? 50? 10?
  - Take a guess!
- Assume we can investigate 1 solution per cpu cycle at a rate of 10 GHz (that’s 10 billion per second). Then, enumerating all Knapsacks with 60 items takes more than 85 years!
- This effect is called a combinatorial explosion.
- If NP ≠ P, it cannot be avoided. However, we can aim at pushing the intractable instance sizes as far as possible – far enough to solve real-world instances. This is what combinatorial optimization is all about!

Implicit Enumeration

- We cannot afford to enumerate all combinations.
- We must try to enumerate the overwhelming part of all combinations implicitly!
- The only way to do this is by intelligent inference.
  - It is usually easy to find a first solution.
  - The core question to ask for an optimization problem is: Can we achieve a better solution?
  - Answering this question is of course NP-complete.
  - Consequently, we have to try to estimate intelligently.

Relaxations

- We can achieve an upper bound on an optimization problem like Knapsack by computing an optimal solution over a larger set of feasible solutions.
- We can allow more solutions by getting rid of some constraints - hopefully in such a way that the relaxed problem is easier to solve.
- This approach is generally called a relaxation.
- The milder the effect of a relaxation on the objective value, the better our estimate!

Linear Relaxation

- The most commonly used relaxation consists in dropping the constraint that variables be integer.
- In Knapsack for instance, we replace $x_i \in \{0,1\}$ by $0 \leq x_i \leq 1$.
- Then, optimizing the relaxed problem calls for solving a linear program – and we know how to optimize LPs quickly! ☺

Relaxations

- What does a relaxation give us?
  - Dominance: If the relaxation value is lower (for minimization: greater) or equal than the best known solution

    ⇒ All solutions with the current prefix are sub-optimal and need not be looked at at all!
  - Optimality: If the relaxation returns a feasible solution for our original problem

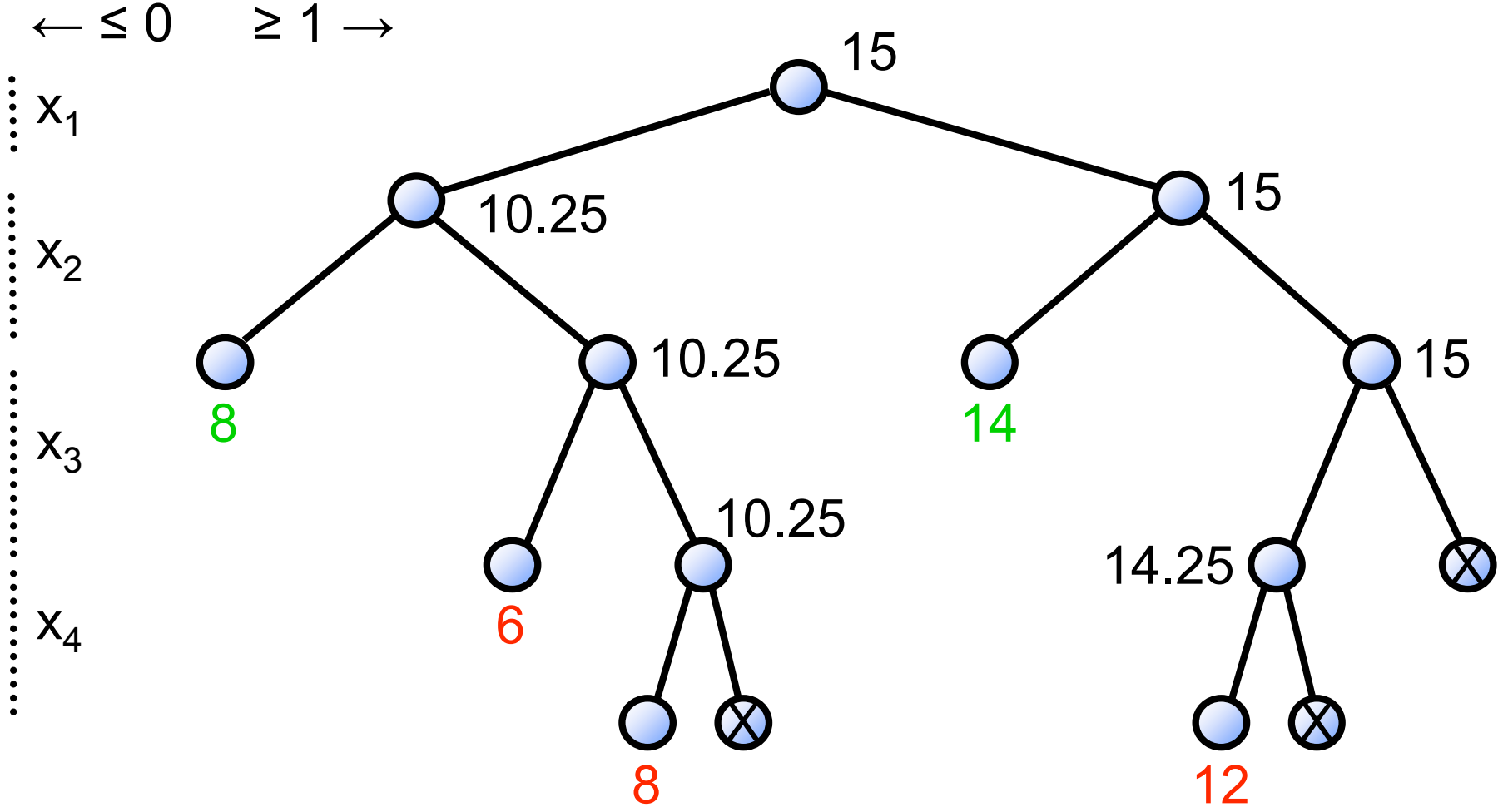
    ⇒ This solution dominates all other feasible solutions, they need not be looked at at all!
  - Infeasibility: If the relaxation is infeasible

    ⇒ There exists no feasible solution with the current prefix, all such combinations need not be looked at at all!
- In all these cases, we are not going to expand the search tree below the current node further ⇒ We prune the search!

Example

- Knapsack Instance
  - Maximize
    - $9\ x_1 + 3\ x_2 + 5\ x_3 + 3\ x_4$
  - such that
    - $5\ x_1 + 2\ x_2 + 5\ x_3 + 4\ x_4 \leq 10$
    - $x_1,x_2,x_3,x_4 \in \{0,1\}$
- LP Relaxation
  - Maximize
    - $9\ x_1 + 3\ x_2 + 5\ x_3 + 3\ x_4$
  - such that
    - $5\ x_1 + 2\ x_2 + 5\ x_3 + 4\ x_4 \leq 10$
    - $0 \leq x_1,x_2,x_3,x_4 \leq 1$

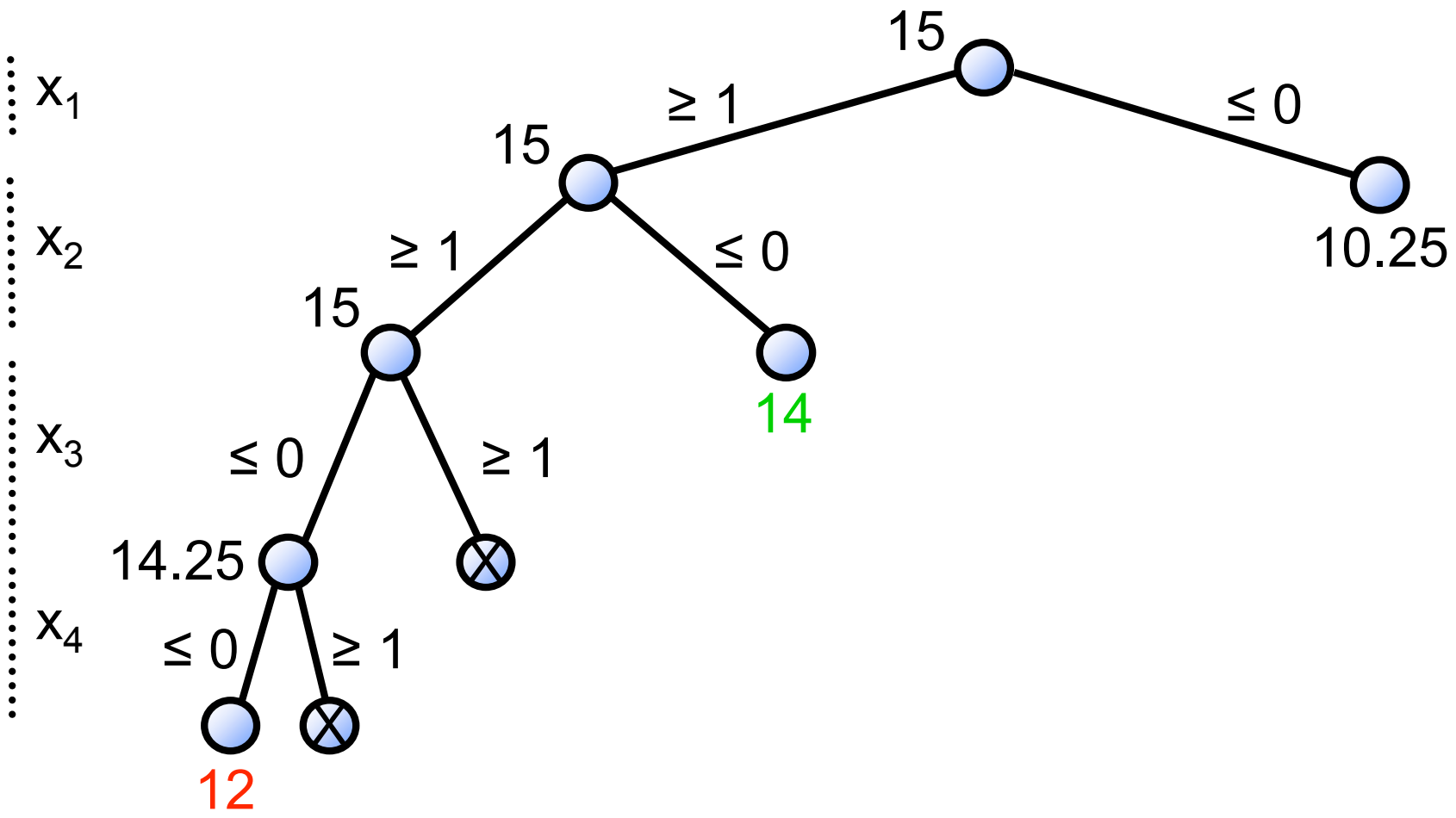
Example



Branching Direction Selection

- In our general Branch-and-Bound scheme, we have some liberty:
  - Which node shall we look at next?
  - Which variable should we branch on?
- We would like to dive into the search tree in order to find a feasible solution (a lower bound) quickly.
- When diving, the question which node to pick next comes down to: which of the two son nodes shall we follow first?

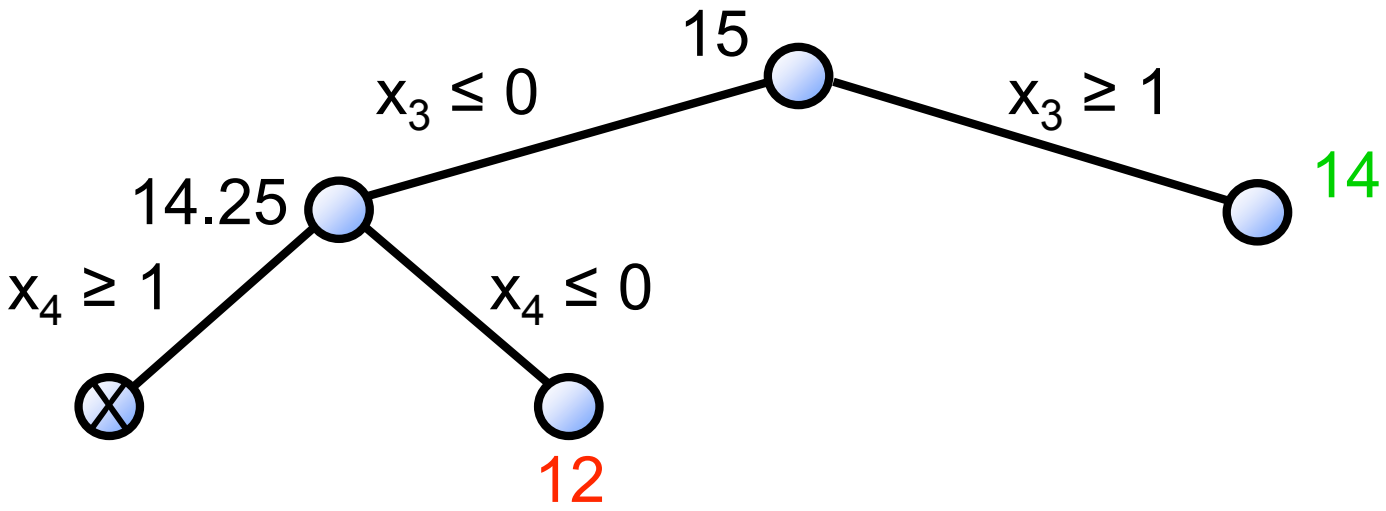
Example



Branching Variable Selection

- In our general Branch-and-Bound scheme, we have some liberty:
  - Which node shall we look at next?
  - Which variable should we branch on?
- In order to have a chance of improving our upper bound, we need to branch on a fractional variable.
- In KP, there is exactly one.

Example



Liberties in B&B

- So far, we took the liberty to select our own branching values and variables.
  - Value selection is a special case of node selection in depth first search.
    - The way how we traverse the search tree is generally determined by our search strategy.
  - Variable selection is a special case of branching constraint selection.
    - Very many different ways to partition the search space are possible.

Search Strategies

- When choosing the next node, we would like:
  - to find a near optimal solution quickly (lower bound improvement in maximization)
  - not to jump too much to make use of incremental data-structures and keep the memory requirements in limits.

Search Strategies

- Depth First Search
  - Finds feasible solutions quickly.
  - Is very memory efficient.
  - Can easily get stuck in sub-optimal parts of the search space.
- Best First Search
  - Look at the node with best relaxation value next.
  - Is provably optimal in the sense that it never visits a node that could be pruned otherwise.
  - A lot of jumping is necessary and memory requirements are prohibitively large (often search degenerates to breadth first search).

Search Strategies

- Depth First Search with Best Backtracking
  - Is a mix of both depth and best first search: perform depth first search until a leaf is found, then backtrack to the node with best relaxation value and so on.
  - Much less jumping than best first search.
  - Is more memory efficient than best first search, but less than DFS – could still be very memory intensive.
- Least Discrepancy Search
  - Follow DFS with heuristic branching direction selection. Investigate leaves in order of increasing discrepancy wrt that heuristic.
  - Memory requirements are within limits.
  - Often finds good solutions early in the search.

Branching Constraint Selection

- When partitioning the search space, we would like:
  - to reduce the relaxation value as quickly as possible (upper bound improvement in maximization)
  - to avoid to double our workload which can happen for example when choosing the wrong branching variable

- The easiest way to partition the search is by branching on one variable.

Branching Constraint Selection

- Unary Branching Constraints
  - Choose the variable which has a fractional part closest to ½.
  - Try to estimate how much enforcing the integrality of a variable will cost at least – degradation method.
  - Follow user-defined priorities.
  - Choose a random variable and combine with restarts.
- Empirically, we prefer balanced search trees over degenerated branches.

Branching Constraint Selection

- In some cases, unary branching constraints cannot achieve balance:
  - $\Sigma\, x_i = 1$, $x_i = 1$ has big, $x_i = 0$ almost no effect!
- Special Ordered Sets
  - SOS-Branching Idea: $\Sigma_{i \in I}\, x_i = 1$ or $\Sigma_{i \notin I}\, x_i = 1$.
  - SOS type 1
    - An ordered set of variables, where at most one variable may take on a nonzero value.
  - SOS type 2
    - An ordered set of variables, where at most two variables may take on nonzero values, and if two variables are nonzero, they must be adjacent in the set.
  - SOS type 3
    - A set of 0-1 variables only one of which may be selected to have the value 1, the other variables in the set having the value 0.

Thank you!