Mathematics behind Linear Equation Systems

CS 149 Staff

September 25, 2008

First, to define what the corners are, we have to get familiar with these concepts behind this.
As defined in the lecture slides, $\kappa(a_1, a_2, ..., a_n)$ is defined to be a set of convex combinations of vectors a_1, a_2, a_3, a_n. For example, when $n = 2$, $\kappa(a_1, a_2)$ contains all points $z = \lambda a_1 + (1 - \lambda)a_2$, which is geometrically the line between two points a_1 and a_2. Moreover, when $n = 3$, $\kappa(a_1, a_2, a_3)$ contains all points in the triangle made by a_1, a_2, and a_3 (including borders).
Now we define convex sets. set $K \subseteq \mathbb{R}^n$ is called **convex**, if and only if $\forall a, b \in K, \kappa(a, b) \subset K$. That is, when you have a set of points, and for all two points in the set, the line between two points has to be included in the set for the set to be called convex. This convex set has some special properties like following:

1. Given a convex set K, a local optimum x_0 is a global optimum for min $c^T x, x \in K$.
   Let $S_{x_0,r} = \{y | \|y - x_0\|^2 \leq r\}$.
   x_0 is local optimum when for some r, $\forall y \in S_{x_0,r} : c^T y \geq c^T x_0$. That is, for every point in the distance r from point x_0, the objective value of that point is greater than that of x_0.
   The question is, for some $x_1 \in K$, can $c^T x_1$ be smaller than $c^T x_0$. To prove, we assume that $c^T x_1 \leq c^T x_0$.
   then we choose $y \in S_{x_0,r} \cap \kappa(x_0, x_1)$; y is on the line of x_0 and x_1 and also in the distance r from x_0.
   $y = \alpha_0 x_0 + \alpha_1 x_1$ where $\alpha_0 + \alpha_1 = 1$.

$$\begin{aligned} c^T y &= c^T(\alpha_0 x_0 + \alpha_1 x_1) && (1) \\ &= \alpha_0 c^T x_0 + \alpha_1 c^T x_1 && (2) \\ &< \alpha_0 c^T x_0 + \alpha_1 c^T x_0 && (3) \\ &= (\alpha_0 + \alpha_1) c^T x_0 && (4) \\ &= c^T x_0 && (5) \end{aligned}$$

   However, since $y \in S_{x_0,r}$, $c^T y \geq c^T x_0$. Therefore, there is a contradiction and there is no such point x_1 s.t. $c^T x_1 < c^T x_0$.

2. Intersection of Convex Sets is convex. You will prove this in Homework 4.

3. Hyperplane is convex

   Let $a \in \mathbb{R}^n, a \neq 0, \alpha \in \mathbb{R}, H = \{x \in \mathbb{R}^n, a^T x = \alpha\}$.
   In these proofs, we use the definition of convex sets.
   We choose $p_1, p_2 \in H$ and $0 \leq \lambda \leq 1$.
   We have to show that $\kappa(p_1, p_2) \subseteq H$.

$$\begin{aligned} a^T(\lambda p_1 + (1-\lambda)p_2) &= \lambda a^T p_1 + (1-\lambda)a^T p_2 && (6) \\ &= \lambda\alpha + (1-\lambda)\alpha && (7) \\ &= \alpha && (8) \end{aligned}$$

   $\therefore \lambda p_1 + (1-\lambda)p_2 \in H$
   $\therefore H$ is convex.

4. Affine vector space is convex Doing similar to the proof above, but with $A_i x = b_i$ instead of $a^T x = \alpha$. We do the same thing for each i, and we will get the result.

5. Halfspace is convex Similarly, from the proof above, we just have to change $=$ to $\geq$.

$$\begin{aligned} a^T(\lambda p_1 + (1-\lambda)p_2) &= \lambda a^T p_1 + (1-\lambda)a^T p_2 && (9) \\ &\geq \lambda\alpha + (1-\lambda)\alpha && (10) \\ &= \alpha && (11) \end{aligned}$$

   $\therefore \lambda p_1 + (1-\lambda)p_2 \in H$
   $\therefore H$ is convex.