Equivalence of LP_S and LP_K

Theorem: LP_S and LP_K are eqivalent problems.

Proof:

Let P_1, c^1 and P_2, c^2 be instances of any optimization problem.
(Refer to Lecture 2, Slide 31 for a definition of optimization problems. Note that we are minimizing).

Lemma: Assume that $\forall x^1 \in P_1 \ \exists x^2 \in P_2$ with $c^{2^T} x^2 \leq c^{1^T} x^1$ and $\forall x^2 \in P_2 \ \exists x^1 \in P_1$ with $c^{1^T} x^1 \leq c^{2^T} x^2$
Then if $x^2 \in P^2$ is optimal, then the analagous $x^1 \in P^1$ is optimal, and they have the same objective function value.

Proof: Let y^1 be any element of P^1. Let y^2 be an element of P^2 with $c^{2^T} y^2 \leq c^{1^T} y^1$ (as in the supposition). Then:

$$c^{1^T} x^1 \leq c^{2^T} x^2 \leq c^{2^T} y^2 \leq c^{1^T} y^1.$$

Therefore x^1 is optimal.$\square$

1. Let LP_K :

$$\min c^T x \ s.t. \ x \geq 0 \ , \ Ax \geq b$$

be given. Then with

$$\tilde{A} := (A, -I), \tilde{b} := b, \tilde{c} := \begin{pmatrix} c \\ 0 \end{pmatrix}, \tilde{x} := \begin{pmatrix} x \\ x^s \end{pmatrix}$$

we can describe an optimization problem in standard form (LP_S) by:

$$\min \tilde{c}^T \tilde{x} \ s.t. \ \tilde{x} \geq 0 \ , \ \tilde{A}\tilde{x} = \tilde{b}$$

Let $\tilde{x}$ be a solution of LP_S. Then x is a solution of LP_K with

$$\tilde{c}^T \tilde{x} = (c^T, 0^T) \cdot \begin{pmatrix} x \\ x^s \end{pmatrix} = c^T x + 0^T \cdot x^s = c^T x$$

Now let x be a solution of LP_K and let $\tilde{x} := \binom{x}{Ax-b}$. Then, with $\tilde{x} \geq 0$ since $x \geq 0$ and $Ax - b \geq 0$, also

$$\tilde{A}\tilde{x} = (A, -I) \cdot \begin{pmatrix} x \\ Ax - b \end{pmatrix} = Ax - Ax + b = b$$

Therefore $\tilde{x}$ is a solution of LP_S. For these it follows that:

$$c^T x = c^T x + 0^T \cdot (Ax - b) = (c^T, 0^T) \cdot \begin{pmatrix} x \\ Ax - b \end{pmatrix} = \tilde{c}^T \tilde{x}$$

2. Let LP_S be given as $\min c^T x \; s.t. \; x \geq 0 \;,\; Ax = b$. Then with

$$\tilde{A} := \begin{pmatrix} A \\ -A \end{pmatrix}, \tilde{b} := \begin{pmatrix} b \\ -b \end{pmatrix}, \tilde{c} := c, \tilde{x} := x$$

   we can describe an optimization problem in canonical form (LP_K) by:

$$\min \tilde{c}^T \tilde{x} \; s.t. \; \tilde{x} \geq 0 \;,\; \tilde{A}\tilde{x} \geq b$$

From these it follows: x is exactly the feasible solution for LP_S when x is a feasible solution for LP_K. Furthermore, it follows that $c^T x = \tilde{c}^T \tilde{x}$. The proof concerning this is similar to the first case.