Problem 1

Your good amigo, who has a small business creating sauce for Tortillas, asks for your advice. He has 300 pounds of tomatoes and 100 pounds of hot peppers. He produces two types of sauces, mild and spicy. For every jar of mild sauce he needs 1 pound of tomatoes and a 1/4 pound of hot peppers. For every jar of spicy sauce he also uses 1 pound of tomatoes, and a 1/2 pound of hot peppers. Moreover, he does not expect to sell more than 250 jars of mild sauce and more than 150 jars of spicy sauce. He can sell each jar of mild sauce for $10 and each jar of spicy sauce for $16.

a) Using the specialized algorithm from class, find the optimal solution. Make sure that you explain your reasoning with each step.

b) Plot the feasibility region on the plane and give the geometric interpretation of each step.

Problem 2

More convexity. Live it. Love it.

a) Show that the set $\{x : x \in \mathbb{R}^2, \|x\|^2 \leq r^2\}$ is convex. Here $\|\cdot\|$ denotes the Euclidean distance:

$$\|x\| = \sqrt{x_1^2 + x_2^2}.$$

Hint: You may use (without proving them) the following two relations:

$$r^2 = r^2(\lambda + 1 - \lambda)^2 = r^2(\lambda^2 + 2\lambda(1-\lambda) + (1-\lambda)^2)$$
$$a_1b_1 + a_2b_2 \leq \max\{a_1^2 + a_2^2, b_1^2 + b_2^2\}.$$

b) Find (and provide proofs) the extreme points of the above set.

c) Let $y = (10, 10)$. Show that the set $\{x : \|x\| \leq 10, \|x - y\| \geq 8\}$ is not convex.

d) **Extra credit:** Prove that $a_1b_1 + a_2b_2 \leq \max\{a_1^2 + a_2^2, b_1^2 + b_2^2\}$.

Problem 3

Find all the basic feasible solutions to the system $\mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 4 & 1 \\ 1 & -1 & 7 & 1 \\ 3 & -2 & 1 & 5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}.$$

Problem 4

Provide all the solutions (in the vector form that we did in class and recitation) of the system $\mathbf{Ax} = \mathbf{b}$, when

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & -4 & 1 & 0 & -3 \\ 0 & 7 & 0 & 1 & -5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}.$$

Problem 5

Consider the following linear program:

$$\begin{aligned} \max \quad & \mathbf{x_1} + \mathbf{x_2} \\ \text{s.t.} \quad & 3\mathbf{x_1} + \mathbf{x_2} \leq 3 \\ & \mathbf{x_1} + 2\mathbf{x_2} \leq 2 \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{1}$$

Recall from lecture that it has the standard form:

$$\begin{aligned} \min \quad & \qquad\quad - \mathbf{x_1} - \mathbf{x_2} \\ \text{s.t.} \quad & \mathbf{s_1} \qquad + 3\mathbf{x_1} + \mathbf{x_2} = 3 \\ & \qquad \mathbf{s_2} + \mathbf{x_1} + 2\mathbf{x_2} = 2 \\ & \mathbf{x} \geq \mathbf{0}, \mathbf{s} \geq 0 \end{aligned} \tag{2}$$

a) Determine the basic feasible solutions corresponding to the bases $B^1 = \{s_1, x_1\}$, $B^2 = \{x_1, x_2\}$.

b) Determine the vectors that give all the solutions to the homogeneous system $\mathbf{Ay} = \mathbf{0}$ for each of the bases obtained in part a (the $\mathbf{y}$'s that we computed in class/recitation).

c) Make a 2D sketch of the problem and draw the vectors you found in part b.

d) Find the optimal solution to the system and formally prove its optimality.