# Problem 1

a) Show how the LP

$$\begin{aligned} \min \quad & \mathbf{c}^T x \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

can be written as

$$\begin{aligned} \min \quad & \mathbf{c}'^T x' \\ \text{s.t.} \quad & \mathbf{A}'\mathbf{x}' \leq \mathbf{b}' \\ & \mathbf{x}' \geq \mathbf{0}. \end{aligned}$$

for the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 77 & 41 & -3 & 27 & 4 \\ 8 & 2 & 9 & -8 & 29 & 9 \\ 23 & 9 & 98 & 9 & 3 & 8 \\ 8 & 6 & 39 & 3 & 1 & 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 14 \\ 18 \\ 19 \\ 2 \end{bmatrix}.$$

.

b) Show how the LP

$$\begin{aligned} \min \quad & \mathbf{c}^T x \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & x_1, x_2, x_4, x_5 \geq 0 \end{aligned}$$

can be written as

$$\begin{aligned} \min \quad & \mathbf{c}'^T x' \\ \text{s.t.} \quad & \mathbf{A}'\mathbf{x}' = \mathbf{b}' \\ & \mathbf{x}' \geq \mathbf{0}. \end{aligned}$$

for the matrices

$$\mathbf{A} = \begin{bmatrix} 3 & 7 & 3 & -3 & 27 & 4 \\ 42 & -12 & 9 & -9 & 29 & 9 \\ 3 & 8 & -8 & 18 & 8 & 4 \\ 0 & 2 & 19 & 3 & 8 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -41 \\ 8 \\ 91 \\ 12 \end{bmatrix}.$$

.

c) In general, how do you convert from

$$\begin{aligned} \min \quad & \mathbf{c}^T x \\ \text{s.t.} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \in \mathbf{R} \end{aligned}$$

to

$$\begin{aligned} \min \quad & \mathbf{c'}^T x' \\ \text{s.t.} \quad & \mathbf{A'x'} = \mathbf{b'} \\ & \mathbf{x'} \geq \mathbf{0}. \end{aligned}$$

# Problem 2

Pigfart Incorporated produces fertilizers that contain nitrate and phosphate in different relative amounts. The amount of phosphate and nitrate in the fertilizer determines the revenue from selling a ton of the fertilizer. There are three different fertilizers that can be produced:

| Name | Nitrate | Phosphate | Revenue per ton sold |
|---|---|---|---|
| Pigfart's Classic (C) | 2% | 3% | \$30,000 |
| Pigfart's Special (S) | 4% | 2% | \$30,000 |
| Pigfart's Best (B) | 4% | 4% | \$45,000 |

The company has a stock of 8 tons nitrate and 6 tons of phosphate that can be used for production.

a) Formulate the problem of maximizing revenue mathematically.

b) Can you sketch the 3−dimensional feasible region graphically?

c) Does your illustration of the problem help to determine how much of each fertilizer should be produced to maximize the revenue?

# Problem 3

In this problem we will familiarize ourselves with the idea of convexity.
Definition of convexity:
A set $K \subseteq \mathbb{R}^n$ is called convex iff for all $a, b \in K : \kappa(a, b) \subseteq K$ where $a, b \in \mathbb{R}^n$ and we define $\kappa(a_1, \ldots, a_m)$ as the set of all convex combinations of $a_1, \ldots, a_m$.
Given $\sum_i \alpha_i = 1$ and $\{a_1, \ldots, a_m\}$ with $\subseteq \mathbb{R}^n$, $\sum_i \alpha_i a_i$ is a convex combination iff $\alpha_1, \ldots, \alpha_m \in \mathbb{R}_{>0}$.

a) You are given a point $y \in \mathbb{R}^n$ $[y = (y_1, \ldots, y_n)]$. Show that the set $\{x : x_i \geq y_i, i = 1, 2, \ldots, n\}$ is convex.

b) You are given a point $y \in \mathbb{R}^n$ and a real $\epsilon > 0$. Show that the set $\{x : |x_i - y_i| < \epsilon, i = 1, 2, \ldots, n\}$ is convex.

## Problem 4

You are given the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 2 & -12 & -9 & 8 & -22 \\ 4 & -3 & 22 & -12 & -6 & 10 & -20 \\ 6 & 15 & -6 & -28 & -27 & 12 & -50 \\ 4 & 0 & 16 & -24 & -17 & 8 & -26 \end{bmatrix},$$

and the vector

$$\mathbf{b} = \begin{bmatrix} -20 \\ -55 \\ 6 \\ -81 \end{bmatrix}.$$

You want to solve the system $\mathbf{Ax} = \mathbf{b}$.

a) Perform Gaussian elimination. You must end up with a system of the form

$$\begin{bmatrix} 1 & 0 & \cdot & 0 & 0 & \cdot & \cdot \\ 0 & 1 & \cdot & 0 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 1 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 1 & \cdot & \cdot \end{bmatrix} \mathbf{x} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}.$$

b) Provide all the solutions to the system $\mathbf{Ax} = \mathbf{b}$.

c) You are now given a cost vector $\mathbf{c}^T = [2\ \ -3\ \ 17\ \ -1\ \ 3\ \ 13\ \ -10]$. What is the cost of every solution that you have computed in part (b)? Note that the cost can be parameterized in $x_3, x_6$ and $x_7$.

d) Find the optimal solution to the linear program

$$\begin{aligned} \min \quad & \mathbf{c}^T x \\ \text{s.t.} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$