The Vickrey-Clarke-Groves Mechanism

CSCI 1440/2440

2025-10-22

We introduce the Vickrey-Clarke-Groves mechanism, a direct mechanism for multiparameter settings, and argue that it is economically efficient and DSIC, assuming independent private values.

1 *Combinatorial Auctions*

After six long weeks of studying single-parameter auctions,[1] we are ready to move on to the study of multiparameter settings. We will, of course, focus our attention on auctions, but there are many other multiparameter settings of interest: e.g., the **task allocation** problem of assigning jobs to workers, where different workers have different abilities to perform the jobs; and the **facility location** problem of locating service centers, such as libraries, knowing that different people live different distances from the proposed locations.

[1] to death!

In single-parameter auctions, there is usually a single good, but even if there is more than one good (e.g., sponsored search), the bidders are characterized by but one parameter. In more general auction environments, there are multiple goods, and bidders are characterized by multiple parameters. In the most general case, bidders cannot be fully characterized except by combinatorially many parameters, one per outcome (i.e., assignment of bundles of goods to bidders). Such auctions are called **combinatorial auctions**.

More formally, when bidders' preferences are combinatorial, we assume they can be described by **valuations**, which are functions from outcomes to values: i.e., $v_i : \Omega \to \mathbb{R}$. Therefore, in direct mechanisms, the focus of the present lecture, bids take the form of *functions* from outcomes to values: i.e., $b_i : \Omega \to \mathbb{R}$.

As usual, we assume quasilinear utilities. Thus, although auction outcomes comprise both an allocation and payment rule, it suffices to consider auction outcomes as allocations only. Specifically, we assume Ω is the space of feasible allocations of goods to bidders.

We also make the following two standard assumptions about bidders' valuations throughout this lecture:

1. Valuations are **normalized**, so that each bidder's value for an outcome in which she is assigned the empty bundle is zero: i.e., $v_i(\omega) = 0$, whenever ω assigns bidder $i \in [n]$ the empty set.
2. Valuations are **monotone**, so that each bidder's values are weakly increasing in the size of her assigned bundle: i.e., $v_i(\omega) \leq v_i(\omega')$ whenever $\omega \subseteq \omega'$, for all $i \in [n]$ and $\omega, \omega' \in \Omega$.

The monotonicity assumption is also called **free disposal**, as it costs the bidders nothing to accrue additional goods.

Note that together these assumptions imply that no bidder's value for any bundle is negative: i.e., for all $i \in [n]$ and for all $\omega \in \Omega$, $v_i(\omega) \geq 0$, since the value of the empty bundle is zero, and accruing additional goods can only increase this value.

2 *Auction Goals*

As usual, we are driven by three auction design goals:

1. incentives: we seek auctions in which reporting bids truthfully is an equilibrium (IC), preferably in dominant strategies (i.e., DSIC), and bidders participate willingly (IR), preferably *ex-post*
2. economic efficiency (e.g., welfare maximization)
3. computational efficiency

It should be immediately apparent that designing such an auction in a computationally efficient manner is a non-trivial endeavor. Just describing a single bidder's valuation—and hence, communicating bids to the auctioneer—can take exponential time and space. We proceed nonetheless, temporarily abandoning the third design goal.

3 *The Vickrey-Clarke-Groves Mechanism*

Recall our recipe for designing IC auctions in a single-parameter setting: 1. collect bids, and then find an allocation that optimizes the designer's objective, given those bids; and 2. tack prices onto that allocation which ensure that the auction outcome is IC and IR.

The first question we ask is, does this design recipe extend to a multiparameter setting? In other words, can we still

1. Solve **winner determination**, i.e., find an economically efficient (e.g., welfare-maximizing) allocation: e.g., given bids $\mathbf{b}$,

   $$\omega^* \in \arg\max_{\omega \in \Omega} \sum_{i \in [n]} b_i(\omega),$$

   and

2. Charge each winner an appropriate payment, so as to ensure the IC and IR properties hold *ex-post*?

The complexity of the winner determination problem can make it very difficult, if not impossible, to deploy auctions designed via this procedure in practice. We return to this issue later.

For the moment, we explore the more philosophical question, is there a payment formula that ensures the IC and IR properties? The answer to this question is indeed yes. The formula is the following:

$$p_i(\omega^*) = h_i(\mathbf{b}_{-i}) - \sum_{j \neq i \in [n]} b_j(\omega^*)$$

for some $h_i : T_{-i} \to \mathbb{R}$, where, as above, ω^* is an efficient allocation, given bids $\mathbf{b}$. Any mechanism that uses this payment formula is called a **Groves** mechanism. Before we unpack this formula, let's prove that the Groves mechanism is both DSIC and *ex-post* IR.

Our proof and the discussion that follows rely on the following notation: Given a bidder i and a bid profile $\mathbf{b}_{-i}$, we write $f(b_i) \doteq f(b_i, \mathbf{b}_{-i})$ to denote an efficient allocation, assuming i reports b_i: i.e.,

$$f(b_i) \in \arg\max_{\omega \in \Omega} \sum_{j \in [n]} b_j(\omega).$$

Bidder i's utility at such an efficient allocation is then

$$u_i(b_i, \mathbf{b}_{-i}) = v_i(f(b_i)) - \left(h_i(\mathbf{b}_{-i}) - \sum_{j \neq i \in [n]} b_j(f(b_i)) \right)$$

Theorem 3.1. *The Groves mechanism is DSIC.*

Proof. Bidder i's utility is given by

$$u_i(b_i, \mathbf{b}_{-i}) = v_i(f(b_i)) + \sum_{j \neq i \in [n]} b_j(f(b_i)) - h_i(\mathbf{b}_{-i}).$$

Note that $h_i(\mathbf{b}_{-i})$ is a constant, independent of bidder i's report. Hence, if bidder i had the power to choose the allocation, she would choose one that maximizes

$$v_i(f(b_i)) + \sum_{j \neq i \in [n]} b_j(f(b_i)).$$

Although she does not have this power, she can maximize this value by reporting truthfully, because the VCG mechanism, which seeks to optimize

$$b_i(f(b_i)) + \sum_{j \neq i \in [n]} b_j(f(b_i)),$$

would thus select an optimizing allocation on her behalf. □

Beyond incentive compatibility, the other important incentive property is *ex-post* individual rationality, i.e., bidder participation in the mechanism should be voluntary.

Another way to state this second objective is that bidders' utilities should always be non-negative, regardless of the outcome. Recall that bidder i's utility is given by:

$$u_i(b_i, \mathbf{b}_{-i}) = v_i(f(b_i)) + \sum_{j \neq i \in [n]} b_j(f(b_i)) - h_i(\mathbf{b}_{-i}).$$

To ensure that this quantity is non-negative, we require

$$v_i(f(b_i)) + \sum_{j \neq i \in [n]} b_j(f(b_i)) \geq h_i(\mathbf{b}_{-i}).$$

We can satisfy the desired inequality by letting $h_i(\mathbf{b}_{-i})$ be the sum total of all bidders' bids, *except bidder i's*, at an efficient allocation: i.e.,

$$h_i(\mathbf{b}_{-i}) = \max_{\omega \in \Omega} \sum_{j \neq i \in [n]} b_j(\omega), \tag{1}$$

This quantity works, because $f(b_i)$ is an efficient allocation, and all bidders' values are non-negative at all outcomes, so adding bidder i into the mix (on the LHS of Equation 3) cannot decrease welfare.

This choice of $h_i(\mathbf{b}_{-i})$ is called the **Clarke pivot rule**. Auctions, or more generally mechanisms, that use the aforementioned design process together with the Clarke pivot rule in the payment formula are called **Vickrey-Clarke-Grove (VCG)** mechanisms.[2]

[2] Vickrey auctions are a special case of VCG mechanisms, the latter of which apply beyond auctions: e.g., in the realm of public goods provisioning.

With the Clarke pivot rule and non-negative valuations, the VCG mechanism is not only DSIC, it is also *ex-post* IR: i.e., participation is voluntary. Moreover, the VCG mechanism never pays bidders to participate (i.e., all payments are non-negative).[3]

[3] Homework 6, Problem 1

In spite of these desirable properties, the VCG mechanism is computationally complex. Moreover, it exhibits some bizarre (and undesirable) behavior. You will explore the following anomalies in this week's homework exercises:

1. The VCG mechanism may generate less revenue for the auctioneer when an additional bidder participates.[4]

[4] Homework 6, Problem 2, Part 1

2. The VCG mechanism may allocate goods to bidders with strictly positive valuations, and still generate zero revenue.[5]

[5] Homework 6, Problem 2, Part 2

3. The VCG mechanism may generate more utility for bidders who collude to submit untruthful bids. Indeed, bidders can collude with themselves by submitting what are called **false-name** bids![6]

[6] Homework 6, Problem 3

4 *Interpreting the VCG Payment Formula*

We now describe two ways to interpret the VCG payment formula.

When bidder i is present, the other bidders, namely $[n] \setminus \{i\}$, may not achieve as much welfare (collectively) as they do when i is not present. With bidder i present, the welfare generated is

$$\omega^* \in \arg\max_{\omega \in \Omega} \sum_{j \in [n]} v_j(\omega),$$

whereas

$$\max_{\omega \in \Omega} \sum_{j \neq i \in [n]} v_j(\omega)$$

is the welfare that would be generated were bidder i not present. The net (i.e., difference in) welfare for the set of bidders $[n] \setminus \{i\}$ is

$$\max_{\omega \in \Omega} \sum_{j \neq i \in [n]} v_j(\omega) - \sum_{j \neq i \in [n]} v_j(\omega^*). \tag{2}$$

This quantity is exactly the payment the VCG mechanism charges bidder i. Thus, we can view bidder i's payment as the **externality** she imposes on all the other bidders, collectively.

Example 4.1. Assume a single-good auction, in which bidder i has the highest value for the good. At the efficient allocation ω^*, the second term $\sum_{j \neq i \in [n]} v_j(\omega^*)$ in Equation 2 equals 0, so that the VCG mechanism charges the winner the first term in Equation 2, namely $\max_{\omega \in \Omega} \sum_{j \neq i \in [n]} v_j(\omega)$, which (not coincidentally!) is the second-highest bid. Thus, the second-price, sealed-bid auction (a.k.a., the Vickrey auction) is the VCG mechanism assuming only one good.

Another way of interpreting the payment formula is in terms of rebates. Expand the payment formula as follows:

$$\begin{aligned} p_i(\omega^*) &= \max_{\omega \in \Omega} \sum_{j \neq i \in [n]} v_j(\omega) - \sum_{j \neq i \in [n]} v_j(\omega^*) \\ &= v_i(\omega^*) + \max_{\omega \in \Omega} \sum_{j \neq i \in [n]} v_j(\omega) - \sum_{j \neq i \in [n]} v_j(\omega^*) - v_i(\omega^*) \\ &= v_i(\omega^*) - \left[\sum_{j \in [n]} v_j(\omega^*) - \max_{\omega \in \Omega} \sum_{j \neq i \in [n]} v_j(\omega) \right]. \end{aligned}$$

With this reformulation, we see that bidder i's payments are precisely her value less a non-negative quantity. This quantity can be understood as the amount of additional welfare that can be attributed to i's presence: i.e., the net value of i, in terms of welfare, to the economy.

Example 4.2. In a Vickrey auction for a single good, suppose, without loss of generality, bidder 1 wins (ω^*), and bidder 2's bid is the second highest. Bidder 1 pays $p_1(\omega^*) = v_1 - (v_1 - v_2) = v_2$.

Here is an example of how the VCG mechanism operates with three bidders and two goods.

Example 4.3. Suppose there are three bidders and two goods, A and B, with valuations as described in Table 1.

The efficient allocation ω^* gives bundle $\{A, B\}$ to bidder 3, and results in total welfare of 4.

1. Without bidder 1, the maximum possible welfare would be 4. Bidder 1's contribution to welfare is zero. Payments for bidder 1 are

$$p_1(\omega^*) = 4 - 4 = 0 - (4 - 4) = 0.$$

Bundle	v_1	v_2	v_3
$\emptyset$	0	0	0
$\{A\}$	2	1	1
$\{B\}$	2	1	1
$\{A, B\}$	2	1	4

Table 1: Bidder valuations for winning subsets of goods in $G = \{A, B\}$.

2. Without bidder 2, the maximum possible welfare would be 4. Bidder 2's contribution to welfare is zero. Payments for bidder 2 are
$$p_2(\omega^*) = 4 - 4 = 0 - (4 - 4) = 0.$$

3. Without bidder 3, the maximum possible welfare would be 3. (We would allocate A to bidder 1 and B to bidder 2.) Payments for bidder 3 are
$$p_3(\omega^*) = 3 - 0 = 4 - (4 - 3) = 3.$$