Sponsored Search

CSCI 1440/2440

2025-09-24

1 *Sponsored Search*

Digital advertising earnings in the U.S. keep reaching new highs. Indeed, Google ads was projected to top the $1 trillion mark in 2024.

A good portion of this revenue is accrued via **sponsored search**, or **position**, auctions, in which advertisement **slots**, or positions, are sold alongside organic search results. We will now explore an auction for selling this online advertisement space, a practical and profitable application, as you can see by the numbers above!

Assume n bidders (online advertisers) are competing for one of k slots on a page that results from a keyword search (e.g., "TV"). Each slot can be allocated to at most one bidder, and each bidder can be allocated at most one slot.

For each slot j, there is an associated probability that a user conducting an organic search clicks on an ad in that slot. This probability is called the **click-through-rate** (CTR).[1] For slot j, we denote the CTR by α_j, and we assume $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$.

[1] In reality, the probability a user clicks on an ad depends on both its position *and* its relevance.

Each bidder i also has a private value v_i that corresponds to how much she values a user clicking on her ad (e.g., an estimate of how much she expects to profit per click). Thus, if a bidder is allocated slot j (i.e., $x_i = \alpha_j$) and pays p_i, her utility is given by $u_i = \alpha_j v_i - p_i$.

We now proceed to design a sponsored search auction, meaning an allocation scheme and an accompanying payment rule, for slots on a web page. The mechanism collects one bid b_i from each bidder $i \in [n]$, and then allocates each slot to at most one bidder and each bidder at most one slot, in an allocation $\mathbf{x}(\mathbf{b})$. Our auction maximizes welfare and satisfies incentive compatibility (so that we can instead write $\mathbf{x}(\mathbf{v})$), individual rationality, and ex-post feasibility. We use Myerson's lemma to argue that it satisfies the incentive constraints.

Welfare Maximization Problem In the sponsored search setting, welfare is the quantity $\sum_i v_i x_i(\mathbf{v})$, where the allocation vector $\mathbf{x}$ contains each of the values $\alpha_1, \ldots, \alpha_k$ at most once, and all other entries are 0. Since the α's are weakly decreasing, this quantity is optimized by first sorting the bidders in weakly decreasing order by value, and then awarding the jth slot to jth bidder in this list,[2] for $1 \leq j \leq k$.

[2] breaking ties randomly

Monotonicity Fix a bidder i and a profile $\mathbf{v}_{-i}$. Figure 1 shows bidder i's allocation as a function of her bid $b \in T$. For example, if i bids between b_j and b_{j-1}, her allocation is α_j. In other words, to be allocated α_j or higher, a bidder must bid at least b_j.

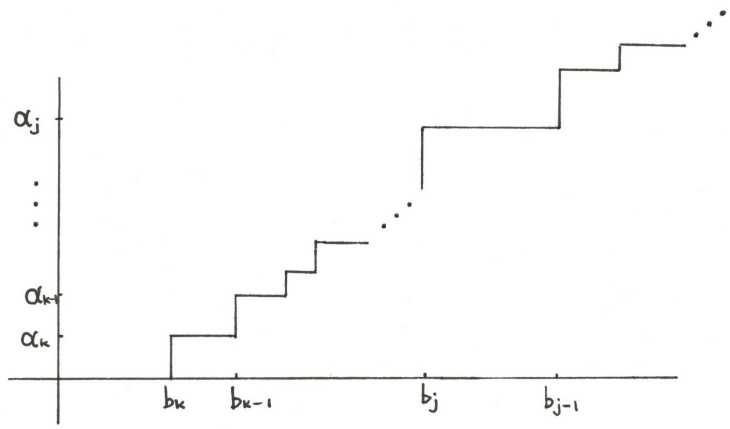

Figure 1: Bidder i's allocation function. (Image courtesy of Zechen Ma.)

Proposition 1.1. *This allocation rule is monotonically weakly increasing.*

Proof. If $b < b_k$, then $x_i(b, \mathbf{v}_{-i}) = 0$, so increasing the bid cannot possibly lower the allocation. Indeed, for all $\epsilon > 0$, $x_i(b+\epsilon, \mathbf{v}_{-i}) \geq x_i(b, \mathbf{v}_{-i}) = 0$. On the other hand, if $b \geq b^*$ is a winning bid, so that $x_i(b, \mathbf{v}_{-i}) = \alpha_j$, for some $j \in \{1, \ldots, k\}$, then for all $\epsilon > 0$, $x_i(b+\epsilon, \mathbf{v}_{-i}) = \alpha_t$, for some $t \in \{1, \ldots, s\}$, because $b_i + \epsilon > b_i$, and the allocation rule ensures that higher bids yield higher CTRs. In other words, since $\alpha_t \geq \alpha_s$, it follows that $x_i(b_i + \epsilon) \geq x(b_i)$. $\square$

Payments The sponsored search allocation rule is "jumpy," meaning piecewise constant on the continuous interval $[0, v_i]$, and discontinuous at points $\{z_1, z_2, \ldots, z_\ell\}$ in this interval. Hence, Myerson payments are as follows (assuming $\alpha_{k+1} = 0$): for $v_i \in (b_j, b_{j-1}]$,

$$\begin{aligned} p_i(v_i, \mathbf{v}_{-i}) &= \sum_{j=1}^{\ell} z_j \cdot [\text{jump in } x_i(\cdot, \mathbf{v}_{-i}) \text{ at } z_j] \\ &= b_j \alpha_j - \sum_{t=k}^{j+1, \text{ by } -1} (b_{t-1} - b_t)\alpha_t \\ &= \sum_{t=k}^{j, \text{ by } -1} (\alpha_t - \alpha_{t+1}) b_t \end{aligned}$$

Additional details:

$$
\begin{aligned}
p_i(v_i, \mathbf{v}_{-i}) &= v_i x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i})\,\mathrm{d}z \\
&= v_i \alpha_j - \left[\int_0^{b_k} 0\,\mathrm{d}z + \int_{b_k}^{b_{k-1}} \alpha_k\,\mathrm{d}z + \int_{b_{k-1}}^{b_{k-2}} \alpha_{k-1}\,\mathrm{d}z + \cdots + \int_{b_{j+1}}^{b_j} \alpha_{j+1}\,\mathrm{d}z + \int_{b_j}^{v_i} \alpha_j\,\mathrm{d}z \right] \\
&= v_i \alpha_j - \left[(b_{k-1} - b_k)\alpha_k + (b_{k-2} - b_{k-1})\alpha_{k-1} + \cdots + (b_j - b_{j+1})\alpha_{j+1} + (v_i - b_j)\alpha_j \right] \\
&= b_j \alpha_j - \left[(b_{k-1} - b_k)\alpha_k + (b_{k-2} - b_{k-1})\alpha_{k-1} + \cdots + (b_j - b_{j+1})\alpha_{j+1} \right] \\
&= b_j \alpha_j - \sum_{t=k}^{j+1,\ \text{by } -1} (b_{t-1} - b_t)\alpha_t
\end{aligned}
$$

A picture is worth a thousand formulas: see Figure 2.

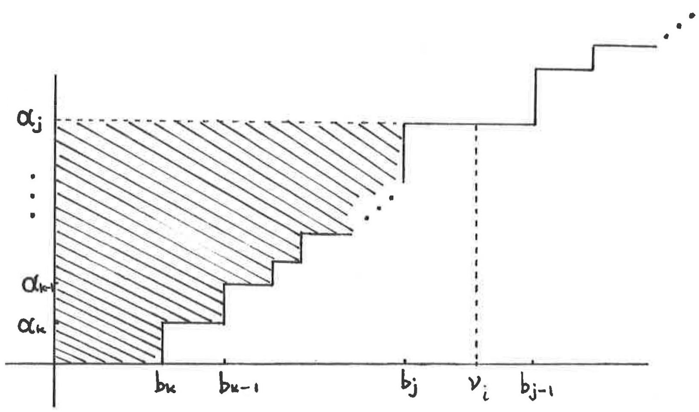

Figure 2: Bidder i's payments for bidding in $[b_j, b_{j-1}]$. (Image courtesy of Zechen Ma.)