Order Statistics and Revenue Equivalence

CSCI 1440/2440

2025-09-10

We derive the first-, second-, and kth-order statistics for the uniform distribution on $[0,1]$. We use these results to prove that the expected revenue of the first- and second-price auctions are equal.

1 *Order Statistics*

Definition 1.1. The **kth-order statistic**, denoted $X_{(k;n)}$, is the the kth-largest value among n i.i.d.draws of a random variable X.

In particular, the first-order statistic is the maximum of n draws, the second-order statistic is the second highest of n draws, and the nth-order statistic is the minimum of n draws.[1]

[1] It is equally legitimate to define order statistics in the reverse order, so that the first-order statistic is the minimum, instead of the maximum, of n draws.

Order statistics are useful in analyzing the outcomes of first- and second-price auctions, as these outcomes depend on the highest and second-highest draws from the distribution of bidders' values.

Consider a random variable X that is uniform on $[0,1]$. If $n = 1$, then we expect the value of the first[2]-order statistic to be the expected value of X itself, namely $1/2$. If $n = 2$, then we would expect the two order statistics to divide the unit interval into thirds, in which case the expected value of the first-order statistic is $2/3$ and the expected value of the second-order statistic is $1/3$. In general, we would expect n order statistics to divide the unit interval into $n+1$ regions, so that the expected value of the smallest order statistic is $1/n+1$ and the expected value of the largest, is $n/n+1$.

[2] and only

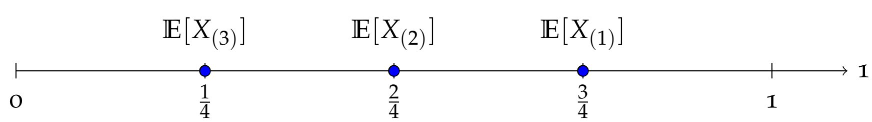

Figure 1: Visualization of the three order statistics for $n = 4$.

Remark 1.2. This result scales accordingly if the random variables are drawn from an arbitrary continuous bounded distribution $[a, b]$, rather than $[0,1]$, so that the expected value of the smallest (resp. largest) order statistic is $a + (b-a)1/n+1$ (resp. $a + (b-a)n/n+1$).

In the rest of this section, we formalize this intuition. We simply write $X_{(k)}$, when n is clear from context. Moreover, we denote the PDF of $X_{(k)}$ by $f_{X_{(k)}}$, and the CDF by $F_{X_{(k)}}$.

1.1 First-Order Statistic

Fix a value of n. We are interested in calculating the expected value of $X_{(1)}$, the first-order statistic, when sampling i.i.d. from a uniform distribution, call it U, on $[0,1]$. That is,

$$\mathbb{E}\left[X_{(1)}\right] = \int_0^1 x f_{X_{(1)}}(x)\ dx.$$

We will proceed by computing the CDF $F_{X_{(1)}}$, which is easy to compute, and then taking derivatives to arrive at the PDF, $f_{X_{(1)}}$.[3]

To derive the CDF, we are interested in the event, $X_{(1)} \leq x$. This event is realized when the highest draw is less than or equal to x, or equivalently, when *all* n draws are less than or equal to x: i.e.,

$$\begin{aligned} F_{X_{(1)}}(x) &= \Pr(X_{(1)} \leq x) \\ &= \Pr(X_j \leq x, \text{ for all } j \in [n]) \\ &= \prod_n \Pr(X \leq x) \\ &= \prod_n F(x) \\ &= x^n. \end{aligned}$$

Now the PDF of $X(1)$ is given by:

$$\begin{aligned} f_{X_{(1)}}(x) &= \frac{d}{dx} F_{X_{(1)}}(x) \\ &= \frac{d}{dx} x^n \\ &= n x^{n-1}. \end{aligned}$$

Therefore, the expected value of the first-order statistic is:

$$\begin{aligned} \mathbb{E}\left[X_{(1)}\right] &= \int_0^1 x f_{X_{(1)}}(x)\ dx \\ &= \int_0^1 n x^n\ dx \\ &= \frac{n}{n+1} x^{n+1} \Big|_0^1 \\ &= \frac{n}{n+1}. \end{aligned}$$

[3] The CDF is defined as follows:

$$F_X(x) = \int_{-\infty}^x f_X(t)\ dt.$$

By the Fundamental Thm of Calculus,

$$f_X(x) = \frac{d}{dx} F_X(x).$$

In particular,

$$f_{X_{(1)}}(x) = \frac{d}{dx} F_{X_{(1)}}(x).$$

1.2 Second-Order Statistic

To derive the CDF of the second-order statistic, we are interested in the event, $X_{(2)} \leq x$. This event is realized when the second-highest draw among n draws is less than or equal to x, which itself can be realized in two ways: either *all* n draws are less than or equal to x (i.e., $X_{(1)} \leq x$) or the highest draw exceeds x, but the remaining $n-1$

draws are all less than or equal to x. The former case happens with probability x^n, while the latter can happen in n different ways, each with probability $x^{n-1}(1-x)$: i.e.,

$$\begin{aligned}
\Pr(X_{(2)} \leq x) &= \Pr(X_{(1)} \leq x) + \sum_{i=1}^{n} \Pr(X_j \leq x, \forall j \neq i \text{ and } X_i > x) \\
&= \Pr(X_{(1)} \leq x) + \sum_{i=1}^{n} \Pr(X_j \leq x, \forall j \neq i) \Pr(X_i > x) \\
&= x^n + nx^{n-1}(1-x).
\end{aligned}$$

Now the PDF of $X(2)$ is given by:

$$\begin{aligned}
f_{X_{(2)}}(x) &= \frac{d}{dx} F_{X_{(2)}}(x) \\
&= \frac{d}{dx} x^n + nx^{n-1}(1-x) \\
&= nx^{n-1} + n(n-1)x^{n-2}(1-x) - nx^{n-1} \\
&= n(n-1)x^{n-2}(1-x).
\end{aligned}$$

Therefore, the expected value of the second-order statistic is:

$$\begin{aligned}
\mathbb{E}\left[X_{(2)}\right] &= \int_0^1 x f_{X_{(2)}}(x) dx \\
&= n(n-1) \int_0^1 \left(x^{n-1} - x^n\right) dx \\
&= n(n-1) \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= n(n-1) \frac{1}{n(n+1)} \\
&= \frac{n-1}{n+1}.
\end{aligned}$$

2 *Revenue Equivalence*

Theorem 2.1. *If bidder's values are uniform i.i.d., then the expected revenue of the first-price auction is equal to that of the second-price auction, assuming bidders behave according to their respective equilibrium strategies.*

Proof. The support of the uniform distribution does not matter; we choose $[0, 1]$ for convenience. Let Rev_1 and Rev_2 denote the expected revenue of the first- and second-price auctions, respectively.

In a second-price auction, the bidder with the highest value wins, paying the second-highest bid, which, because the auction is truthful, is in fact the second-highest value. Therefore, the expected revenue is equal to the expected second-highest value, which is precisely the expected value of the second-order statistic: i.e.,

$$\text{Rev}_2 = \frac{n-1}{n+1}.$$

In a first-price auction, the winner is the bidder with the highest value, and she pays her bid. Therefore, the expected revenue is the expected value of the highest bid. What is this expected value?

Recall our assumption that the bidders' values are uniformly distributed in the range $[0,1]$. Since the first-price equilibrium bid is $({}^{n-1}/_{n})\,v$, assuming value v, it follows that these bids are distributed in the range $[0, {}^{n-1}/_{n}]$. Moreover, since the equilibrium bid function is weakly increasing in value, and the bids are the values times a constant scale factor, the bids are likewise uniformly distributed.

The expected revenue is therefore the expected value of the first-order statistic (i.e., the highest bid), assuming n draws from a uniform distribution on $[0, {}^{n-1}/_{n}]$: i.e., the first-order statistic on the uniform $[0,1]$ distribution scaled by ${}^{n-1}/_{n}$:

$$\text{Rev}_1 = \left(\frac{n}{n+1}\right)\left(\frac{n-1}{n}\right) = \frac{n-1}{n+1}$$

Therefore, $\text{Rev}_1 = \text{Rev}_2$. □

Remark 2.2. This revenue equivalence result holds for any **standard** auction, meaning any auction that allocates to a highest bidder, so that equilibrium bids are weakly increasing in value, assuming 1. bidders' values drawn from a continuous, bounded distribution (e.g., uniform on $[a,b]$, for some $a, b \in \mathbb{R}$), and 2. the lowest-type bidder pays some constant value (usually 0) at equilibrium. Any auction that charges any arbitrary combination of the bids is a standard auction, even with arbitrary additive fees (e.g., the winner pays the average of the first- and second-highest prices plus $10).

A *kth-Order Statistic*

Beta Function The Beta function $\text{B}(x,y)$ is by the following integral:

$$\text{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dy.$$

When x and y are positive integers, this function simplifies as:

$$\text{B}(x,y) = \frac{(x-1)!(y-1)!}{(x+y-1)!}.$$

We will use the Beta function in (the very last step of) our derivation of the expected value of the kth-order statistic.

To start, let's compute the probability the kth-order statistic lies in some small interval $[x, x+\Delta x] \subset [0,1]$. When the draws are i.i.d.,

$$\Pr(X_{(k)} \in [x, x+\Delta x]) = n\binom{n-1}{k-1} \Pr(X < x)^{n-k} \Pr(X \in [x, x+\Delta x]) \Pr(X > x+\Delta x)^{k-1} + \text{O}(\Delta x^2).$$

The middle three probabilities are, respectively, the chance of:

- exactly $n-k$ values less than x,
- exactly one value between x and $x+\Delta x$, and
- exactly $k-1$ values greater than $x+\Delta x$.

This gives the probability of one specific arrangement of this form, so we multiply by the number of possible arrangements. There are n possible agents who could have a value between x and $x+\Delta$, after which there are $\binom{n-1}{k-1}$ possible groups of agents who could have values greater than x, after which the remaining $n-k$ agents are fixed. **N.B.** There is also a chance that multiple values fall between x and $x+\Delta x$. As each such probability will contain a Δx^i term with $i \geq 2$, we include the term $\mathrm{O}(\Delta x^2)$.

The assumption that X is uniformly distributed on $[0,1]$ yields the following further simplification:

$$\Pr(X_{(k)} \in [x, x+\Delta x]) = n\binom{n-1}{k-1} x^{n-k}\, \Delta x\, (1-x-\Delta x)^{k-1} + \mathrm{O}(\Delta x^2).$$

Letting $x_{i+1} = x_i + \Delta x$, we can express the expectation of interest in discretized space as follows:

$$\sum_{i=1}^{m} x_i \;\Pr(X_{(k)} \in [x_i, x_{i+1}]).$$

To calculate the corresponding continuous expectation, we take the limit as $m \to \infty$, so that the Δx terms become arbitrarily small:

$$\begin{aligned}
\mathbb{E}\left[X_{(k)}\right] &= \lim_{m\to\infty} \sum_{i=1}^{m} x_i \;\Pr(X_{(k)} \in [x_i, x_i+\Delta x]) \\
&= n\binom{n-1}{k-1}\left(\lim_{m\to\infty} \sum_{i=1}^{m} x_i^{n-k+1}\, \Delta x\, (1-x_i-\Delta x)^{k-1} + \mathrm{O}(\Delta x^2)\right) \\
&= n\binom{n-1}{k-1}\int_0^1 x^{n-k+1}(1-x)^{k-1}dx \\
&= n\binom{n-1}{k-1}\mathrm{B}(n-k+2, k) \\
&= \left(\frac{n!}{(k-1)!(n-k)!}\right)\left(\frac{(n-k+1)!(k-1)!}{(n+1)!}\right) \\
&= \frac{n-(k-1)}{n+1}.
\end{aligned}$$