

# The Prophet Inequality

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We describe the prophet inequality, and a relatively simple, near-optimal auction, which follows as an immediate consequence.

## 1 The Prophet Inequality

The prophet inequality<sup>1</sup> is a surprising result that lower bounds the expected reward<sup>2</sup> one can obtain in the following game: You are the leader of an investment team. Your team is in possession of some assets that your most trustworthy monetary advisor claims will be worthless after  $n$  days. Panicked, you decide to liquidate these assets within the next  $n$  days, but want to do so while still maximizing your rewards from selling the assets. Fortunately, your quant team comprises CSCI 1440/2440 alumni who are accurately able to back out the *exact* distributions  $F_i$  of the assets' market values  $\pi_i$  on day  $i$ , for all  $1 \leq i \leq n$ .<sup>3</sup> Equipped with this information, it is your job to make the executive decision as to when you will sell the assets.

Once again:

1. The market value  $\pi_i$  on day  $i$  is drawn from some distribution  $F_i$ .
2. At the start of day  $i$ , you observe the market value  $\pi_i$ .
3. You then have two choices: cash out now, or wait.
4. If you cash out now, you walk away with a reward of  $\pi_i$ .
5. If you wait, then you can no longer sell your assets at price  $\pi_i$ , and you will face the same decision the next day.
6. After  $n$  days, your assets become worthless.
7. Your task is to devise a strategy for deciding whether you should sell on day  $i$  or wait another day, after observing  $\pi_i$ .

This problem is an instance of an **optimal stopping problem**, where the goal is to devise a **stopping rule** that tells you when to *stop*, which in our setup means when to cash out. Another famous stopping problem is the **secretary problem**. There, you are interviewing secretaries,<sup>4</sup> and your goal is to decide when to stop interviewing so as to maximize the probability of hiring the best secretary.

Optimal stopping rules can be computed via dynamic programming, when  $n$  is small. For certain variants of the problem, these

<sup>1</sup> Ulrich Krengel and Louis Sucheston. On semiamarts, amarts, and processes with finite value. In J. Kuelbs, editor, *Probability on Banach Spaces*, volume 4, pages 197–266. Dekker, 1978; and Ester Samuel-Cahn. Comparison of threshold stop rules and maximum for independent nonnegative random variables. *The Annals of Probability*, 12(4):1213–1216, 11 1984

<sup>2</sup> or utility, or payoff, or etc.

<sup>3</sup> Perhaps surprisingly, market values from one day to the next in this game are independent of one another.

<sup>4</sup> or applicants for any job, really, including spouse

rules take the form: reject the first  $m$  applicants, and then accept the first applicant thereafter who is preferable to the first  $m$ .

In this lecture, we will not derive an optimal stopping rule, but rather an approximately-optimal one. Our rule will take the form of a thresholding strategy, where we accept the first applicant whose value is above a pre-computed threshold. Note that computing this threshold depends on knowledge of the distributions  $F_i$  of the random variables. Implicit in the  $F_i$  notation, it should be clear that we are also making another strong assumption, namely independence.

Imagine a clairvoyant *prophet* who can see all of the  $\pi_i$ 's in advance. With this information in hand, the optimal strategy is straightforward: choose the largest  $\pi_i \geq 0$ . In what follows, we will be comparing the expected reward of a very simple thresholding strategy to that of the prophet. The simple thresholding strategy is: fix a value  $t$ , and then sell the assets on the first day, if any, that  $\pi_i \geq t$ .<sup>5</sup>

We claim that for an appropriate choice of  $t$ , this simple thresholding strategy is a 2-approximation of the optimal: i.e., the prophet's strategy. In fact, there are two distinct choices of  $t$ , both of which achieve this goal. We derive one of them presently—the one which lends itself to the development of approximately-optimal auctions.

<sup>5</sup> Note that this strategy is clearly sub-optimal, as you would never retain the assets on the very last day; nevertheless, our simple strategy never sells the assets if  $\pi_i$  never exceeds  $t$ .

## 2 Proof of the Prophet Inequality

Let APX and OPT denote the expected reward of the thresholding strategy and the prophet, respectively. We will proceed by first lower bounding APX, and then upper bounding OPT.

The assets are sold on day  $i$ , at the threshold  $t$ , only if  $\pi_i \geq t$  and  $\pi_j < t, \forall j < i$ . Furthermore, the assets are *not* sold after all  $n$  days only if  $\pi_j < t, \forall 1 \leq j \leq n$ . We let  $p_t$  denote this probability: i.e.,  $p_t = \mathbb{E} [\mathbb{1}\{\pi_j < t, \forall 1 \leq j \leq n\}]$ . Using this notation, the probability that the assets are sold on one of days 1 through  $n$ , namely  $\mathbb{E} [\sum_{i=1}^n \mathbb{1}(\pi_i \geq t; \pi_j < t, \forall j < i)]$ , is  $1 - p_t$ .

Now, APX, the expected reward of the threshold strategy, is the sum of the expected values that the assets are sold on each day  $i$ : i.e.,

$$\begin{aligned} \text{APX} &= \mathbb{E} \left[ \sum_{i=1}^n \pi_i \mathbb{1}\{\text{assets are sold on day } i\} \right] \\ &= \sum_{i=1}^n \mathbb{E} [\pi_i \mathbb{1}\{\text{assets are sold on day } i\}] \\ &= \sum_{i=1}^n \mathbb{E} [(t + \pi_i - t) \mathbb{1}\{\text{assets are sold on day } i\}] \\ &= t \mathbb{E} \left[ \sum_{i=1}^n \mathbb{1}\{\pi_i \geq t; \pi_j < t, \forall j < i\} \right] + \sum_{i=1}^n \mathbb{E} [(\pi_i - t) \mathbb{1}\{\pi_i \geq t; \pi_j < t, \forall j < i\}] \end{aligned}$$

$$\begin{aligned}
&= t(1 - p_t) + \sum_{i=1}^n \mathbb{E} [(\pi_i - t) \mathbb{1}\{\pi_i \geq t; \pi_j < t, \forall j < i\}] \\
&= t(1 - p_t) + \sum_{i=1}^n \mathbb{E} [(\pi_i - t) \mathbb{1}\{\pi_i \geq t\} \mathbb{1}\{\pi_j < t, \forall j < i\}] \\
&= t(1 - p_t) + \sum_{i=1}^n \mathbb{E} [(\pi_i - t) \mathbb{1}\{\pi_i \geq t\}] \mathbb{E} [\mathbb{1}\{\pi_j < t, \forall j < i\}] \\
&= t(1 - p_t) + \sum_{i=1}^n \mathbb{E} [(\pi_i - t)^+] \mathbb{E} [\mathbb{1}\{\pi_j < t, \forall j < i\}] \\
&\geq t(1 - p_t) + \sum_{i=1}^n \mathbb{E} [(\pi_i - t)^+] \mathbb{E} [\mathbb{1}\{\pi_j < t, \forall 1 \leq j \leq n\}] \\
&= t(1 - p_t) + \sum_{i=1}^n \mathbb{E} [(\pi_i - t)^+] \Pr(\pi_j < t; \forall 1 \leq j \leq n) \\
&= t(1 - p_t) + \sum_{i=1}^n \mathbb{E} [(\pi_i - t)^+] p_t
\end{aligned}$$

Having established a lower bound on APX, we proceed to derive an upper bound on OPT. Since the prophet will sell the assets when their price is maximal, and will not sell them for a negative reward, the value of the prophet's strategy can be upper bounded as follows:

$$\text{OPT} = \mathbb{E} \left[ \max_i \pi_i \right] = \mathbb{E} \left[ \max_i \pi_i^+ \right]$$

Finally, since the lower bound on APX holds for all thresholds  $t$ , we choose  $t$  such that  $p_t = 1/2$  to obtain the desired 2-approximation:

$$\begin{aligned}
\text{APX} &\geq \sum_{i=1}^n \mathbb{E}[(\pi_i - t)^+] p_t + t(1 - p_t) \\
&= 1/2 \left( \sum_{i=1}^n \mathbb{E}[(\pi_i - t)^+] + t \right) \\
&= 1/2 \left( \mathbb{E} \left[ \sum_{i=1}^n (\pi_i - t)^+ \right] + t \right) \\
&\geq 1/2 \left( \mathbb{E} \left[ \max_i (\pi_i - t)^+ \right] + t \right) \\
&\geq 1/2 \left( \mathbb{E} \left[ \max_i (\pi_i^+ - t) \right] + t \right) \\
&= 1/2 \mathbb{E} \left[ \max_i \pi_i^+ \right] \\
&= 1/2 \text{OPT}
\end{aligned}$$

The only non-trivial step in this derivation is the third to last, which follows as  $\max(\pi_i - t, 0) = \max(\pi_i, t) - t \geq \max(\pi_i, 0) - t$ .

It turns out that another threshold value works just as well as the choice we just derived. This value is  $t = 1/2 \mathbb{E} [\pi^*]$ , where  $\pi^* =$

$\max_i \pi_i$ . A particularly elegant proof of this fact appears in Correa *et al.*,<sup>6</sup> as their proof implies both our choice and this latter choice at once. In particular, they prove  $\text{APX} \geq p_t t + (1 - p_t)(\mathbb{E}[\pi^*] - t)$ . So, both choosing  $t$  such that  $p_t = 1/2$  (as above) and setting  $t = 1/2 \mathbb{E}[\pi^*]$  yield the desired inequality.

<sup>6</sup> José R. Correa, Patricio Foncea, Ruben Hoeksma, Tim Oosterwijk, and Tjark Vredeveld. Recent developments in prophet inequalities. *SIGecom Exchanges*, 17(1):61–70, 2018

### 3 Applications of the Prophet Inequality

The prophet inequality can be used to design a 2-approximation of Myerson’s optimal auction. The key idea is to take the  $i$ th bidder’s virtual value to be the value of the  $i$ th asset.

Assume  $n$  bidders, with values drawn from regular distributions  $F_1, \dots, F_n$ , and  $\varphi_i$  as bidder  $i$ ’s virtual value function. This auction works as follows: Set the reserve price for bidder  $i$  to be  $\varphi_i^{-1}(t)$ , where  $t$  is determined by the prophet inequality (i.e., choose  $t$  s.t. the good is sold with probability  $1/2$ , and then allocate to any one of the (possibly multiple) bidders who meet their reserve.<sup>7</sup>

Any bidder  $i$  who meets their reserve bids such that  $v_i \geq \varphi_i^{-1}(t)$ ; equivalently,  $\varphi_i(v_i) \geq t$ . Therefore, the revenue APX accrued by an auctioneer that employs a bidder-dependent threshold strategy with threshold  $\varphi_i^{-1}(t)$  is lower bounded by  $1/2 \text{OPT}$ , where OPT is the revenue of Myerson’s optimal auction:

$$\begin{aligned} \text{APX} &= \mathbb{E} \left[ \sum_{i=1}^n \varphi_i(v_i) \mathbb{1}\{\text{bidder } i \text{ wins}\} \right] \\ &\geq 1/2 \left( \mathbb{E} \left[ \max_i (\varphi_i(v_i))^+ \right] \right) \\ &= 1/2 \text{OPT} \end{aligned}$$

<sup>7</sup> To ensure this auction is DSIC, this allocation rule must be monotonic, and then payments must be determined accordingly.

This auction, while sub-optimal, is simpler than Myerson’s auction, as it relies on the bidders’ virtual value functions only to set their reserve prices. It is also more natural than Myerson’s auction, as it can (and probably should; see Sidenote 7) allocate to the highest bidder who clears her reserve, charging that bidder the bid of the second-highest bidder who clears his reserve.

There is also a very straightforward connection between the prophet inequality and posted-price mechanisms. Imagine a (match-box) car salesman who has posted a price on his prized Maserati. Each day, another potential buyer enters his store, admires the car, and buys it or not, depending on their value for the car. They buy it precisely when their value exceeds (or matches) the price.

Assuming the salesman plans to entertain exactly  $n$  buyers, one per day, and has knowledge of their value distributions, namely  $F_1, \dots, F_n$ , he can post  $\varphi_i^{-1}(t)$  as the price for buyer  $i$ , where as above,

$t$  is determined by the prophet inequality and  $\varphi_i$  is buyer  $i$ 's virtual value function. By the argument above, this posted price will guarantee him half the revenue of Myerson's optimal auction, assuming the buyers walk away with the car if their value exceeds their price.

As the salesman operates in a sequential setting, perhaps he can earn even more revenue by dynamically updating his threshold strategy. You will explore this question in your homework.

### *References*

- [1] José R. Correa, Patricio Foncea, Ruben Hoeksma, Tim Oosterwijk, and Tjark Vredeveld. Recent developments in prophet inequalities. *SIGecom Exchanges*, 17(1):61–70, 2018.
- [2] Ulrich Krengel and Louis Sucheston. On semiamarts, amarts, and processes with finite value. In J. Kuelbs, editor, *Probability on Banach Spaces*, volume 4, pages 197–266. Dekker, 1978.
- [3] Ester Samuel-Cahn. Comparison of threshold stop rules and maximum for independent nonnegative random variables. *The Annals of Probability*, 12(4):1213–1216, 11 1984.