Posted-Price Mechanisms: Approximating Revenue

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We show that simple mechanisms can generate near-optimal revenue.

1 *A Simple Mechanism*

Unlike the basic first- and second-price auctions (without a reserve), which always produce a winner, and hence the potential for revenue, the posted-price mechanism provides no such guarantee. If the posted price is larger than the highest bidder's value, then no one will win, and accordingly, there will be no revenue. So how can a seller determine an optimal posted-price?

When distributional knowledge is available, the seller can potentially solve analytically for an optimal posted price. But this approach is not robust to incorrectly specific distributional information. An alternative is to develop a mechanism that is approximately optimal for an unknown distribution, regardless of that distribution. We call such mechanisms **prior independent**, and aim to develop such mechanisms in this lecture.

The mechanism we will analyze is simple, namely "post a random price," by sampling from the distribution F. We call the expected revenue of this mechanism $\mathrm{APX}(F)$: i.e.,

$$\mathrm{APX}(F) = \mathop{\mathbb{E}}_{\pi \sim F}[\mathrm{Rev}(\pi)]$$

Just how well can this mechanism do? That is, what is the ratio of $\mathrm{APX}(F)$ to $\mathrm{OPT}(F)$? We will develop some machinery in this lecture that will enable us to answer this question. The machinery is a bit complex, but with it, the answer to the question is simple.

Here is an outline of the required machinery:

1. We redefine revenue in terms of quantiles, which leads to a simple geometric interpretation of $\mathrm{APX}(F)$.

2. Next, we prove that the virtual value function is the derivative of the revenue curve.

3. Then we assume regularity, which implies that this derivative is weakly increasing in values, and likewise weakly decreasing in quantiles. Under this assumption, the revenue curve is concave.

2 *Expected Revenue in Quantile Space*

Recall that revenue is the product of the posted price and the probability of a sale, where the probability of a sale is the probability that a draw from F exceeds π. But this latter probability is precisely what a quantile represents! That is, at a given quantile q, the probability of a sale is simply q. Moreover, the value of a sale at quantile q is $v(q) = F^{-1}(1-q)$. In other words,

$$\text{Rev}(q) = F^{-1}(1-q)\, q,$$

Next, let's investigate expected revenue in quantile space:

$$\mathop{\mathbb{E}}_{q \sim U[0,1]} [\text{Rev}(q)] = \int_0^1 \text{Rev}(q)\, g(q)\, \mathrm{d}q = \int_0^1 \text{Rev}(q)\, \mathrm{d}q$$

The first step in this derivation follows from the definition of revenue, while the second step follows from the fact that quantiles are necessarily uniformly distributed, so that $g(q) = 1$. In words, *in quantile space, the expected revenue is the area under the revenue curve.*

With this observation in mind, let's now investigate the expected revenue of our mechanism in quantile space. To do so, we will analyze an equivalent version of the mechanism, which rather than drawing and posting a random price from F, draws a random quantile from $U(0,1)$, and then posts price $v(q)$.

Proposition 2.1. *$APX(F)$ (i.e., the expected revenue of posting a random price) is the area under the revenue curve in quantile space.*

Proof. First observe the following:

$$\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}\pi} &= \frac{\mathrm{d}}{\mathrm{d}\pi}(1 - F(\pi)) \\ &= -f(\pi). \end{aligned}$$

Equivalently, $\mathrm{d}q = -f(\pi)\,\mathrm{d}\pi$. Now,

$$\begin{aligned} \text{APX}(F) &= \mathop{\mathbb{E}}_{\pi \sim F} [\text{Rev}(\pi)] \\ &= \int_{\underline{v}}^{\overline{v}} \text{Rev}(\pi)\, f(\pi)\, \mathrm{d}\pi \\ &= -\int_1^0 \text{Rev}(q)\, \mathrm{d}q \\ &= \int_0^1 \text{Rev}(q)\, \mathrm{d}q \\ &= \mathop{\mathbb{E}}_{q \sim U[0,1]} [\text{Rev}(q)] \end{aligned}$$

The third line follows not only because $\mathrm{d}q = -f(\pi)\,\mathrm{d}\pi$, but further, when we integrate from low values to high in value space, we likewise integrate from high quantiles to low in quantile space. □

In sum, we have expressed $\mathrm{APX}(F)$ in quantile space as expected revenue. Equivalently, $\mathrm{APX}(F)$ is the area under the revenue curve.

3 *Properties of Revenue in Quantile Space*

We have shown that expected revenue in quantile space is the area under the revenue curve. But if the revenue curve is arbitrarily complex, it may be difficult to compute this integral. We now set out to show that the revenue curve cannot be arbitrarily complex; on the contrary, in quantile space it is always concave, assuming F is regular.

3.1 *Virtual Values*

For starters, we show how virtual values relate to the revenue curve. Let's differentiate the revenue curve w.r.t. quantile q:

$$\begin{aligned}\frac{\mathrm{dRev}(q)}{\mathrm{d}q} &= \frac{\mathrm{d}\left(qF^{-1}(1-q)\right)}{\mathrm{d}q} \\ &= [q]'[F^{-1}(1-q)] + [q][F^{-1}(1-q)]' \\ &= F^{-1}(1-q) + [q][F^{-1}(1-q)]'.\end{aligned}$$

To differentiate the function inverse, we use the chain rule:

$$\frac{\mathrm{d}z}{\mathrm{d}y} = \frac{\mathrm{d}z}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}y}.$$

More specifically, for a function $g(h(y))$, let $z = g(x)$ and $x = h(y)$:

$$\frac{\mathrm{d}g(h(y))}{\mathrm{d}y} = \frac{\mathrm{d}g(x)}{\mathrm{d}x}\frac{\mathrm{d}h(y)}{\mathrm{d}y}.$$

Now choose $g = F$ and $h = F^{-1}$, so that $g(h(y)) = F(F^{-1}(y)) = y$. Taking the derivatives of both sides of this equation then yields:

$$1 = \frac{\mathrm{d}y}{\mathrm{d}y} = \frac{\mathrm{d}F(F^{-1}(y))}{\mathrm{d}y} = \frac{\mathrm{d}F(x)}{\mathrm{d}x}\frac{\mathrm{d}F^{-1}(y)}{\mathrm{d}y} = f(x)[F^{-1}(y)]'.$$

Rearranging,

$$[F^{-1}(y)]' = \frac{1}{f(x)} = \frac{1}{f(F^{-1}(y))}.$$

Thus, by a second application of the chain rule,

$$[q][F^{-1}(1-q)]' = \frac{-q}{f(F^{-1}(1-q))},$$

from which it follows that

$$\frac{\mathrm{dRev}(q)}{\mathrm{d}q} = F^{-1}(1-q) + \left(\frac{-q}{f(F^{-1}(1-q))}\right).$$

Now, since $q(v) = 1 - F(v)$ and $v = F^{-1}(1 - q(v))$, we conclude that

$$\begin{aligned}\frac{\mathrm{dRev}(q)}{\mathrm{d}q} &= v - \frac{1 - F(v)}{f(v)} \\ &= \varphi(v).\end{aligned}$$

In other words, the derivative of the revenue curve in quantile space, also called the **marginal revenue**, is the virtual value function!

3.2 *Concavity*

Next, we prove that the revenue curve in quantile space is concave, assuming F is regular. In other words, we assume the virtual value function in *value* space is weakly increasing, or equivalently, the virtual value function in *quantile* space is weakly decreasing.

As an example, if values are uniformly distributed on $[0, 1]$, the virtual value function $\varphi(v) = 2v - 1$ is weakly increasing in values. Likewise, since $v = F^{-1}(1 - q) = 1 - q$, the virtual value function $\varphi(q) = 1 - 2q$ is weakly *decreasing* in quantiles.

Definition 3.1 (Concave function)**.** A real-valued function f is weakly concave if, for any x, y in the domain, and for any $c \in [0, 1]$,

$$f((1 - c)x + cy) \geq (1 - c)f(x) + cf(y).$$

Equivalently, for any x, y in the domain,

$$f\left(\frac{x + y}{2}\right) \geq \frac{f(x) + f(y)}{2}.$$

Remark 3.2. You can understand concavity by graphing f. Draw a line from point $(x, f(x))$ to $(y, f(y))$. The function f is concave if it lies above the line in the interval $[x, y]$, for all choices of x and y.

Proposition 3.3. *Assuming regularity, the revenue curve in quantile space is weakly concave.*

Proof. We show that the revenue curve is weakly concave using a bit of calculus. Let $q_1 \leq q_2$, so that $v(q_1) \geq v(q_2)$.

Integrating the virtual value function from quantile q_1 to q_2 yields:

$$\int_{q_1}^{q_2} \varphi(v(q))\,\mathrm{d}q = \mathrm{Rev}(q)\Big|_{q_1}^{q_2} = \mathrm{Rev}(q_2) - \mathrm{Rev}(q_1).$$

It follows that

$$\begin{aligned}\int_{q_1}^{\frac{q_1+q_2}{2}} \varphi(v(q))\,\mathrm{d}q &= \mathrm{Rev}\left(\frac{q_1 + q_2}{2}\right) - \mathrm{Rev}\left(q_1\right) \\ \int_{\frac{q_1+q_2}{2}}^{q_2} \varphi(v(q))\,\mathrm{d}q &= \mathrm{Rev}\left(q_2\right) - \mathrm{Rev}\left(\frac{q_1 + q_2}{2}\right).\end{aligned}$$

By the regularity assumption, the virtual value function is weakly decreasing in quantiles. Hence,

$$\int_{q_1}^{\frac{q_1+q_2}{2}} \varphi(v(q))\,\mathrm{d}q \geq \int_{\frac{q_1+q_2}{2}}^{q_2} \varphi(v(q))\,\mathrm{d}q$$

$$\mathrm{Rev}\left(\frac{q_1+q_2}{2}\right) - \mathrm{Rev}\,(q_1) \geq \mathrm{Rev}\,(q_2) - \mathrm{Rev}\left(\frac{q_1+q_2}{2}\right)$$

$$\mathrm{Rev}\left(\frac{q_1+q_2}{2}\right) \geq \frac{\mathrm{Rev}(q_2)+\mathrm{Rev}(q_1)}{2}.$$

We conclude that the revenue curve is weakly concave. □

Since the derivative of the revenue curve in quantile space is the virtual value function, which, by regularity, is assumed to be weakly decreasing in quantiles, we can arrive at this same conclusion by showing that the integral of any weakly decreasing function (i.e., virtual value) is a weakly concave function (i.e., revenue).

Theorem 3.4. *Let g be a positive integrable function defined for all $x \in [a,b]$. Consider a function G defined by $G(x) = \int_a^x g(t)dt$. If g is weakly decreasing on $[a,b]$, then G is weakly concave on that interval.*

Proof by picture. Consider a weakly decreasing function g on interval $[a,b]$, such as the one depicted in Figure 1. Consider as well an arbitrary point $x_0 \in [a,b]$ and an arbitrary $\delta > 0$.

The value of G at x_0 is equal to the gray area in Figure 1. The black area is the incremental area corresponding to $x_0 + \delta$, and the blue area is the further incremental area corresponding to $x_0 + 2\delta$.

Since g is positive, the value of G at $x_0 + \delta$ (both the gray and the black areas), must exceed the value of G at x_0 (only the gray area); likewise for the value of G at $x_0 + 2\delta$ relative to the value of G at $x_0 + \delta$. So G is increasing. Moreover, since g is weakly decreasing, the blue area is no larger than the black area. These observations ensure that every line segment joining arbitrary points on G lies entirely below G. So G is concave. (See Figure 2.) □

Corollary 3.5. *Assuming regularity, the revenue curve in quantile space is weakly concave.*

Proof. Recall that the derivative of the revenue curve in quantile space is the virtual value function. Hence, the revenue curve is the integral of the virtual value function. By regularity, the virtual value function is weakly decreasing in quantiles. □

4 *Posted-Price Mechanisms*

We now return to our regularly scheduled program. Our goal is to derive an approximation ratio for the simple mechanism, "post a

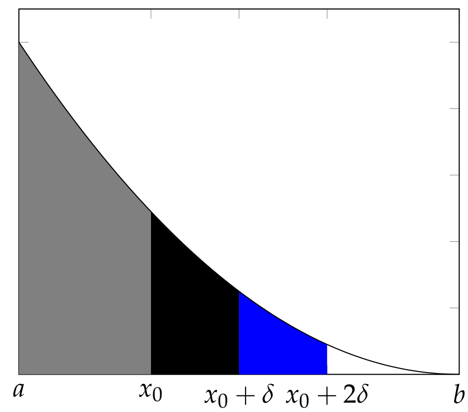

Figure 1: Decreasing function $g(x)$, where $x \in [0, 1]$.

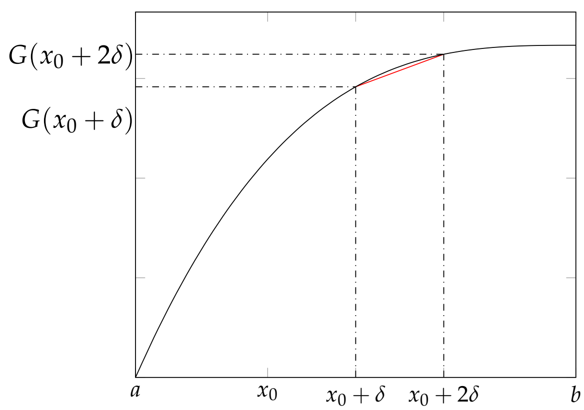

Figure 2: Concave function $G(x) = \int_0^x g(t)dt$, where $x \in [0, 1]$.

random price," in the single bidder setting, assuming F is regular. That is, we seek to calculate $\text{APX}(F)/\text{OPT}(F)$.

Recall that $\text{APX}(F)$ equals the area under the revenue curve. To complete our analysis, let π^* denote the optimal posted price, so that $\text{Rev}(\pi^*) = \text{OPT}$. Analogously, let $q^* = 1 - F(\pi^*)$ be the quantile corresponding to the optimal posted price π^*, so that $\text{Rev}(q^*)$ also equals $\text{OPT}(F)$. We can depict this latter quantity by drawing a box of height $\text{Rev}(q^*)$ and width 1, as shown in Figure 3.

As is evident in this figure, $\text{OPT}(F)$ upper bounds $\text{APX}(F)$. To lower bound $\text{APX}(F)$, observe that the area under the revenue curve is at least the area of the triangle with vertices $(0, 1)$, $(1, 0)$, and $(q, \text{Rev}(q^*))$ (see Figure 3). The area of this triangle is half the area of the box, and hence half the value of $\text{OPT}(F)$.

Therefore, posting a price that is simply a random draw from F, yields, in expectation, at least half the total expected revenue of the optimal posted-price mechanism: i.e., $\text{APX}(F) \geq 1/2\, \text{OPT}(F)$.

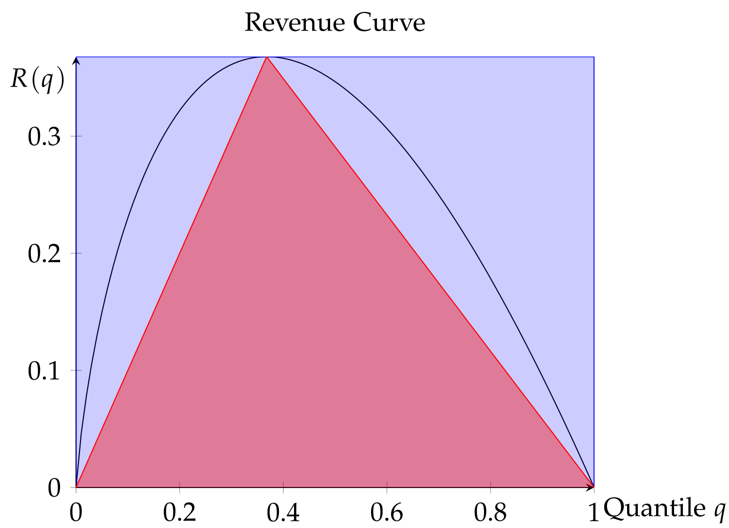

Figure 3: Revenue curve of the exponential distribution, $\lambda = 1$. The blue box represents the expected revenue generated by the optimal auction. The red triangle represents a lower bound on the expected revenue generated by posting a random price drawn from F.

4.1 *A Prior-**De**pendent Mechanism*

Assume an infinite supply[1] of copies of some good: e.g., a digital good, such as an audio or video recording.

[1] It suffices that there exist more supply than demand.

Assume further that there are multiple potential bidders for this good, each of whom draws its value from the distribution F (i.e., buyers' values are i.i.d. draws from F). What is the total expected revenue of posting a price randomly drawn from F?

Using our earlier analysis, we can expect to generate at least half the optimal revenue from each individual bidder, and since values are i.i.d., we conclude that this mechanism, in the infinite-supply setting, yields an approximation ratio of $1/2$.

4.2 *A Prior-Independent Mechanism*

We end this lecture by describing a prior-independent mechanism, again for the case of infinite supply, that achieves this same approximation ratio of $1/2$, assuming symmetric bidders, meaning bidders values are drawn i.i.d. The mechanism works as follows:

1. Collect a sealed bid from each bidder.
2. Select a bidder j uniformly at random.
3. Remove bidder j from the mechanism.
4. Set as a reserve price v_j, for all bidders $i \neq j$.
5. Allocate to all bidders that meet this reserve, and charge them v_j.

First, observe that this, being a posted-price mechanism, is DSIC. No bidder can improve her utility by bidding non-truthfully.

Second, how well does this mechanism do? Let APX denote the total expected revenue of this mechanism, and let OPT denote the total expected revenue of the optimal mechanism (i.e., posting the monopoly price—which requires knowledge of F).

The way this mechanism selects a reserve price v_j is equivalent to drawing a random value from some (unknown) distribution. But now at most $n-1$ bidders can pay, so the approximation ratio is:

$$\frac{APX}{OPT} \geq \frac{1}{2}\left(\frac{n-1}{n}\right).$$

We can improve the approximation ratio by offering bidder j a reserve price equal to the value some other bidder $i \neq j$ submits. With this modification, bidder j's contribution to total expected revenue is the same as all the other bidders', and we recover the original approximation ratio of 1/2.

The single-sample second-price *auction* chooses a reserve price in the same way, and then allocates the good to a highest bidder who bids above this reserve, charging her the higher of the second-highest bid and this reserve. This prior-independent auction also achieves an approximation ratio of 1/2,[2] via a similar (geometric) analysis.

[2] See Homework 6, Problem 5.

A The "If" Direction

Proposition 3.3 is not unique to the revenue curve. It applies to all weakly concave functions. Indeed:

Proposition A.1. *A function is weakly concave only if its derivative is weakly decreasing.*

How about the "if" direction? If a function is weakly concave, is its derivative weakly decreasing? It turns out that this is indeed the case.

Proposition A.2. *A function is weakly concave if and only if its derivative is weakly decreasing.*

We don't prove this fact formally here. We simply provide a few examples that shed some light on the situation.

Example A.3 (A strictly concave function with a strictly decreasing derivative)**.** Let f be

$$f(x) = -\frac{x^2}{2} + x.$$

The derivative of f is

$$f'(x) = -x + 1.$$

Both f and f' are shown in Figure 4.

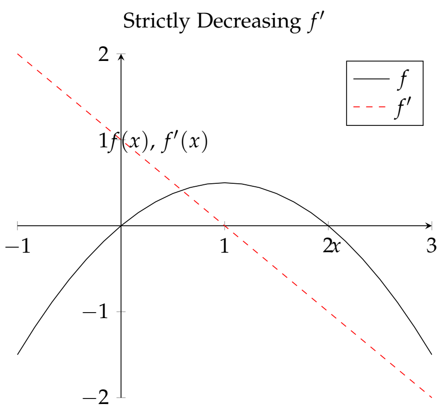

Figure 4: A function f and its strictly decreasing derivative. In this example, there exists a unique global maximizer, $x = 1$.

Example A.4 (A weakly concave function with a weakly decreasing derivative)**.** Let f be

$$f(x) = \begin{cases} -\frac{x^2}{2} - x, & \text{if } x \leq -1 \\ 0.5, & \text{if } x \in [-1, 2] \\ -\frac{x^2}{2} + 2x, & \text{otherwise.} \end{cases}$$

The derivative of f is

$$f'(x) = \begin{cases} -x - 1, & \text{if } x \leq -1 \\ 0, & \text{if } x \in [-1, 2] \\ -x + 2, & \text{otherwise} \end{cases} .$$

Both f and f' are shown in Figure 5.

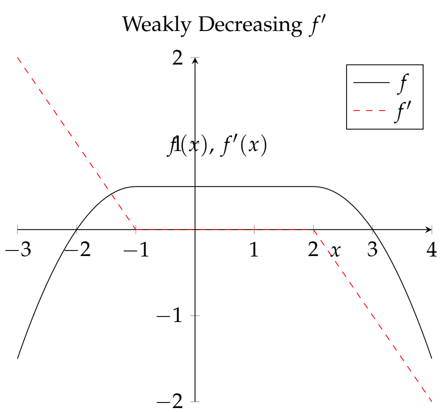

Figure 5: A function f and its weakly decreasing derivative. In this example, there exists multiple global maximizers x, where $x \in [-1, 2]$.

B Sample Revenue Curves in Value and Quantile Space

Sample revenue curves are plotted in value and quantile space in Figures 6 through 9. The area under the curves plotted in quantile space is the expected revenue.

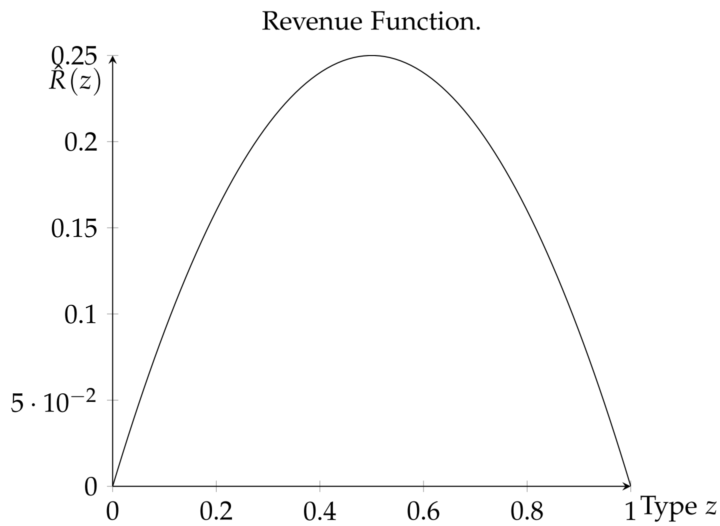

Figure 6: Revenue function of the uniform distribution, plotted in value space.

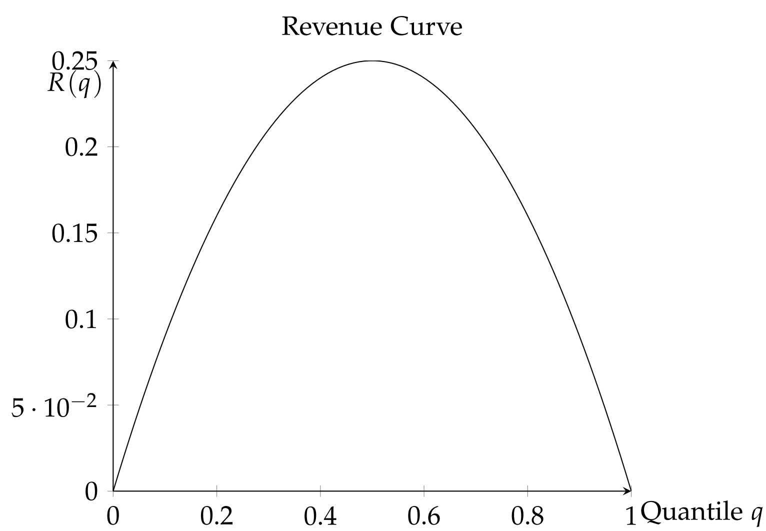

Figure 7: Revenue curve of the uniform distribution, plotted in quantile space.

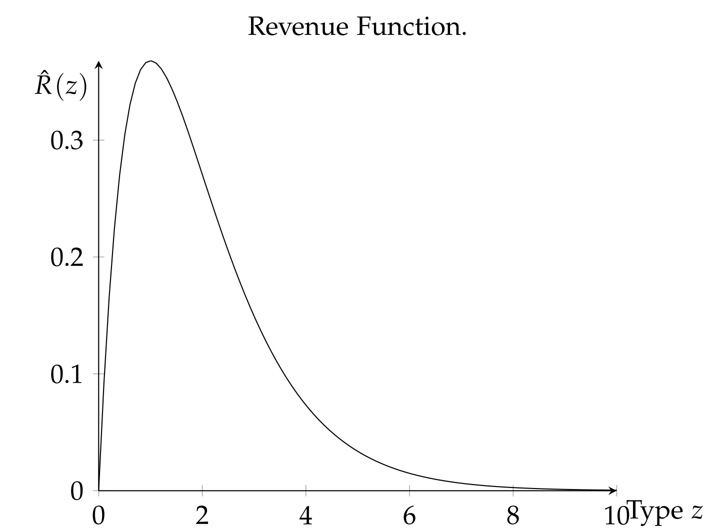

Figure 8: Revenue function of the exponential distribution, $\lambda = 1$, plotted in value space.

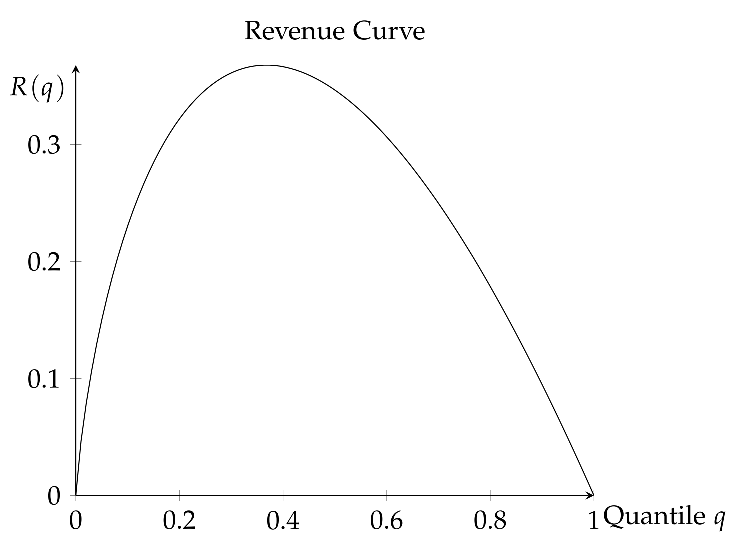

Figure 9: Revenue curve corresponding to the exponential distribution, $\lambda = 1$, plotted in quantile space.